DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

May 11, 2020

9:00 a.m. – 2:00 p.m.

- Today's exam (Part I) contains 5 problems on 2-7 pages. Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed two extra hours, so the total time allotted for today is five hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem on a new page. Put your name and the question number on each page.
- When you do not have time to put answers into forms that satisfy you, indicate ***specifically*** how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem ___" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

Contents

I.1: General Plasma Physics [25 points]

Consider a magneto-electric particle trap in the region $-L < z < L$. To accomplish this trap, suppose a magnetic field in the z direction such that

$$B = \begin{cases} B_0 \left(1 + (R-1)(\frac{z}{L_m})^2\right), & \text{if } -L_m < z < 0; \\ B_0, & \text{if } z \geq 0. \end{cases}$$

Suppose also an electric potential

$$\phi = \begin{cases} 0, & \text{if } z < 0; \\ \phi_0 \left(\frac{z}{L_e}\right)^2, & \text{if } 0 \leq z < L_e; \\ \phi_0, & \text{if } z > L_e. \end{cases}$$

(a) [**4 points**] Describe how ions might be trapped in this configuration of magnetic and electric fields. Would electrons also be trapped in the same fields?

(b) [**6 points**] Derive a trapping condition for confined particles in terms of the particle midplane perpendicular energy $W_{\perp 0}$ and midplane parallel energy $W_{\| 0}$, where these energies are defined at the axial location $z = 0$.

(c) [**2 points**] Sketch the trapping condition in $W_{\perp 0} - W_{\| 0}$ space.

(d) [**2 points**] If trapped ions of charge state q were scattered in pitch-angle, but not in energy, through collisions, from what end of the device would they leave? How does this answer depend on the midplane energy coordinates $W_{\perp 0}$ and $W_{\| 0}$? Please explain very briefly (in one sentence).

(e) [**5 points**] Suppose now that the electric potential is a varying function of time. Show that the second adiabatic invariant can be put in the form

$$W_{\| 0}^{1/2} (z_M + z_E) = \text{const}.$$

Here z_M and z_E are the turning points in the regions $z < 0$ and $z > 0$ respectively. What are z_M and z_E in terms of the parameters L_e, L_m, R, $W_{\perp 0}$, and $W_{\| 0}$. Define $W_c \equiv q\phi_0/(R-1)$. Show that, if $W_{\perp 0}/W_c \sim O(1)$, then $L_e \gg L_m$ implies $z_e \gg z_m$.

(f) [**6 points**] Suppose that the length $L_e(t)$ slowly changes in time, but assume that $L_e(t) \gg L_m$ for all t. Show that, if $L_e(t)$ is slowly shortened from $t = 0$ to $t = t_0$, such that $L_e(0)/L_e(t_0) = \alpha > 1$, then there is a region in $W_{\perp 0} - W_{\| 0}$ space (where coordinates are given at $t = 0$), such that any ions in that region will escape on a different side of the trap by the time $t = t_0$, than they otherwise would have eventually escaped by rare but finite pitch angle scattering had the trap potential not been altered ($\alpha = 1$). Show that this region is triangular in shape with area

$$A \simeq \frac{1}{2}(q\Phi_0)^2 \left(1 - \alpha^{-4/3}\right)^2.$$

Not so helpful hint: You may wish to use (but you do not really need it) the integral $\int_0^1 ((1 - s^2)^{1/2} = \pi/4$.

I.2: Experimental Methods [50 points]

A planar probe is immersed into a weakly collisional steady-state plasma with Maxwellian electron energy distribution function, cold ions, $T_e \gg T_i$:

(a) [**15 points**] Derive the expression for the floating potential of the probe with respect to the plasma assuming no electron emission from the probe.

(b) [**10 points**] Consider the floating probe is heated by plasma to temperatures when it starts to emit electrons. Assume that the temperature of emitted electrons is negligible compared to the temperature of plasma electrons. Derive the expression for the floating potential of the electron emitting probe with respect to the sheath-pre-sheath edge.

(c) [**15 points**] Show and compare qualitatively, the profiles of the electric potential between the probe and the plasma, without the electron emission and with a strong electron emission. Assume a strong electron emission from the wall, i.e., when the coefficient of the electron emission is about 1.

What are the spatial scales (sizes) of the sheath and the pre-sheath regions compared to the Debye length?

(d) [**10 points**] Consider a sweeping bias voltage is applied to the probe. The bias voltage is swept to get the full probe $V - I$ characteristic. Show qualitative changes of this characteristic induced by the electron emission due to the probe heating - show probe $V - I$'s at different probe temperatures (i.e. different electron emission currents).

I.3: General Phenomena [40 points]

Consider the Vlasov-Poisson system for electrostatic perturbations in a hydrogen plasma,

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{q}{m} \nabla \phi \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{1}$$

$$\nabla^2 \phi = -4\pi q \int f d^3 \boldsymbol{v} - e n_0. \tag{2}$$

Here, $q = -e$ is the charge of electrons, f is the distribution of electrons in phase space and $n_0 = \text{const.}$ is the density of the motionless background ions.

(a) [**15 points**] Define

$$Q \equiv \int \left[\int \frac{1}{2} m v^2 f d^3 \boldsymbol{v} + \int G(f) d^3 \boldsymbol{v} + \frac{1}{8\pi} (\nabla \phi)^2 \right] d^3 \boldsymbol{x}.$$

Prove that Q is a constant of motion, i.e.,

$$\frac{dQ}{dt} = 0. \tag{3}$$

Here, $v = |\boldsymbol{v}|$ and $G(f)$ is any well-behaved function of f.

(b) [**3 points**] What is the physical meaning of Eq. (3) when $G(f) = 0$?

(c) [**4 points**] Does the Vlasov-Poisson system (1)-(2) admit a local energy conservation law in the form of

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \mathscr{P} = 0?$$

why?

(d) [**10 points**] Consider a small amplitude (linear) perturbation of the system relative to a homogeneous equilibrium specified by

$$f_0 = f_0(v), \;\; \phi_0 = 0.$$

Let

$$\delta f = f - f_0, \;\; \delta \phi = \phi.$$

Using the fact Q is a constant of motion (3), show that

$$\int \frac{-1}{\frac{\partial f_0}{\partial H}} (\delta f)^2 \, d^3 \boldsymbol{v} d^3 \boldsymbol{x} + \frac{1}{8\pi} \int (\nabla \delta \phi)^2 \, d^3 \boldsymbol{x} = \text{const.}, \tag{4}$$

where $H \equiv \frac{1}{2} m v^2$. (*Hint: Taylor-expand Q for small δf and $\delta \phi$ and select a proper $G(f)$ such that the first order Q vanishes. If you can show it without using $Q = \text{const.}$, that will be fine too.*)

(e) [**8 points**] Using Eq. (4), show that if $f_0(H)$ is a monotonically decreasing function of H, all linear perturbations of the system are stable.

I.4: Irreversible Processes [50 points]

This problem concerns a particular Chapman–Enskog expansion of the ion kinetic equation for a magnetized, weakly collisional plasma. Mathematical formulae of possible utility are provided at the end of the problem.

To answer all of the following questions, you will need the Vlasov–Landau kinetic equation governing the time evolution the ion distribution function f_i. Written in a frame co-moving with the ions' mean velocity $\boldsymbol{u}_i = \boldsymbol{u}_i(t, \boldsymbol{r})$, that equation is

$$\frac{\partial f_i}{\partial t} + (\boldsymbol{u}_i + \boldsymbol{w}) \cdot \boldsymbol{\nabla} f_i + \left(\frac{\nabla p_i}{m_i n_i} + \frac{\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i}{m_i n_i} - \boldsymbol{w} \cdot \nabla \boldsymbol{u}_i + \boldsymbol{w} \times \Omega_i \hat{\boldsymbol{b}} \right) \cdot \frac{\partial f_i}{\partial \boldsymbol{w}}$$
$$= C_{ii}[f_i] + C_{ie}[f_i] + \frac{\boldsymbol{R}_{ie}}{m_i n_i} \cdot \frac{\partial f_i}{\partial \boldsymbol{w}}, \qquad (1)$$

where $\boldsymbol{w} \equiv \boldsymbol{v} - \boldsymbol{u}_i$ is the "peculiar" velocity relative to the mean velocity, $f_i = f_i(t, \boldsymbol{r}, \boldsymbol{w})$, and $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}/B$ is the unit vector in the direction of the magnetic field $\boldsymbol{B} = \boldsymbol{B}(t, \boldsymbol{r})$. The other symbols have their usual meanings: m_i is the ion mass, n_i is the ion number density, $p_i = n_i T_i$ is the (isotropic) ion thermal pressure with $T_i \equiv m_i v_{\mathrm{th}i}^2/2$ being the ion temperature and $v_{\mathrm{th}i}$ being the ion thermal speed, $\boldsymbol{\Pi}_i$ is the ion viscous stress tensor, and $\Omega_i \equiv ZeB/m_i c$ is the ion Larmor frequency with Ze being the ion charge. The collision operator on the right-hand side of equation (1) takes into account both ion–ion collisions (C_{ii}) and ion–electron collisions (C_{ie}), each occurring a rate proportional to their respective (mass-dependent!) collision frequencies, ν_{ii} and ν_{ie}. The final term on the right-hand side accounts for the friction force on the ion fluid due to collisions with electrons, denoted $\boldsymbol{R}_{ie}$. To simplify the ensuing calculations, assume Maxwellian electrons with mass $m_e \ll m_i$ so that, to leading order in the mass ratio,

$$C_{ie}[f_i] + \frac{\boldsymbol{R}_{ie}}{m_i n_i} \cdot \frac{\partial f_i}{\partial \boldsymbol{w}} = \nu_{ie} \frac{\partial}{\partial \boldsymbol{w}} \cdot \left(\boldsymbol{w} f_i + \frac{T_e}{m_i} \frac{\partial f_i}{\partial \boldsymbol{w}} \right), \qquad (2)$$

where T_e is the electron temperature. You are also given that C_{ii} and C_{ie} satisfy Boltzmann's H theorem and that, in field-aligned coordinates in which $\boldsymbol{w} = w_\parallel \hat{\boldsymbol{b}} + \boldsymbol{w}_\perp$,

$$\frac{\partial}{\partial \boldsymbol{w}} = \hat{\boldsymbol{b}} \frac{\partial}{\partial w_\parallel} + \frac{\boldsymbol{w}_\perp}{w_\perp} \frac{\partial}{\partial w_\perp} - \frac{\boldsymbol{w} \times \hat{\boldsymbol{b}}}{w_\perp^2} \frac{\partial}{\partial \vartheta},$$

where ϑ is the gyrophase.

(a) [15 points] Order the dimensionless parameters that appear in equation (1) as follows:

$$\frac{u_i}{v_{\mathrm{th}i}} \sim \sqrt{\frac{m_e}{m_i}} \sim \left| \frac{T_e}{T_i} - 1 \right| \sim \frac{\rho_i}{L} \sim \frac{\lambda_{ii}}{L} \equiv \epsilon \ll 1, \qquad (3)$$

where $\rho_i \equiv v_{\mathrm{th}i}/\Omega_i$ is the ion Larmor radius, $\lambda_{ii} \equiv v_{\mathrm{th}i}/\nu_{ii}$ is the ion–ion collisional mean free path, and L is the characteristic scale of the macroscopic gradients in the plasma. Expand the ion distribution function as $f_i = f_{i0} + \epsilon f_{i1} + \epsilon^2 f_{i2} + \dots$ and write down the lowest order in ϵ at which each term of equation (1) enters relative to $\nu_{ii} f_{i0}$ (e.g., $C_{ii}[f_i]$ enters at $\mathcal{O}(1)$ relative to $\nu_{ii} f_{i0}$). Clearly justify each of your orderings.

- Problem continued on next page -

(b) [5 points] Use your answer to (a) to write down an equation that is valid at $\mathcal{O}(1)$. Solve this equation to show that

$$f_{i0} = f_{\mathrm{M},i} \equiv \frac{n_i}{\pi^{3/2} v_{\mathrm{th}i}^3} \exp\left(-\frac{w^2}{v_{\mathrm{th}i}^2}\right), \quad \text{where} \quad w = |\boldsymbol{v} - \boldsymbol{u}_i| \quad \text{and} \quad v_{\mathrm{th}i}^2 \equiv \frac{2T_i}{m_i}. \tag{4}$$

What constraints must f_{i1} and f_{i2} satisfy so that the parameters n_i, $\boldsymbol{u}_i$, and T_i in equation (4) are the true time- and space-dependent number density, mean velocity, and temperature of the ions?

If you could not do part (a), then provide physical arguments why f_{i0} ought to be gyrotropic and Maxwellian, and answer the above question about constraints.

(c) [5 points] Use your answer to (a) to write down an equation that is valid at $\mathcal{O}(\epsilon)$. In what way(s) is your equation different from that obtained at $\mathcal{O}(\epsilon)$ in Braginskii's expansion for the ions? Why?

If you could not do part (a), then explain how the ordering (2) *differs from Braginskii's.*

(d) [10 points] Solve your $\mathcal{O}(\epsilon)$ equation for f_{i1}. To keep things relatively simple, let $C_{ii}[f_i] = -\nu_{ii}(f_i - f_{\mathrm{M},i})$ and adopt the subsidiary ordering

$$\frac{\rho_i}{\lambda_{ii}} \sim \frac{\nu_{ii}}{\Omega_i} \sim \frac{L_\perp}{L_\parallel} \ll 1, \tag{5}$$

where $L_\perp$ ($L_\parallel$) is the characteristic scale of the macroscopic gradients oriented across (along) the local magnetic-field direction.

Hint: Split f_{i1} into its gyro-averaged part, $\langle f_{i1}\rangle_\vartheta$, and its gyrophase-dependent part, $\widetilde{f}_{i1}$.

(e) [15 points] Use your answer to (d) to compute the leading-order expression for the ion heat flux $\boldsymbol{q}_i$. Briefly explain what each component of $\boldsymbol{q}_i$ represents physically.

If you could not do part (d), then state how you would compute $\boldsymbol{q}_i$ given f_{i1}, and provide physical arguments that anticipate the form of $\boldsymbol{q}_i$.

Possibly useless information:

$$\langle \boldsymbol{w}\rangle_\vartheta = w_\parallel \hat{\boldsymbol{b}}, \quad \langle \boldsymbol{w}\boldsymbol{w}\rangle_\vartheta = w_\parallel^2 \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \frac{w_\perp^2}{2}(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}), \quad \frac{\partial \boldsymbol{w}_\perp}{\partial\vartheta} = -\boldsymbol{w}\times\hat{\boldsymbol{b}}, \quad \frac{\partial(\boldsymbol{w}\times\hat{\boldsymbol{b}})}{\partial\vartheta} = \boldsymbol{w}_\perp,$$

$$(\mathsf{I}\times\hat{\boldsymbol{b}})\cdot\nabla = \hat{\boldsymbol{b}}\times\nabla, \quad \text{where } \mathsf{I} \text{ is the unit dyadic.}$$

$$\int_0^\infty \mathrm{d}x\, x^k\, \mathrm{e}^{-x} \equiv \Gamma(k+1) \ \big(= k! \text{ for integer } k \geq 0\big)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}, \quad \Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$$

I.5: Applied Math Quickie [15 points]

Develop a leading order (in ϵ) uniform approximation to the non-linear differential equation

$$\epsilon \frac{d^2y}{dx^2} + 3\left(\tan(x)\right)^2 \left(\frac{dy}{dx}\right)^2 - \left(\tan(x)\right)^2 y^4 = 0 \tag{1}$$

with the boundary conditions $y(0) = 0$ and $y(\sqrt{3}) = 1$, using the layer matching for $0 < \epsilon \ll 1$. Assume the layer at $x = 0$ and no rescaling needed for y.

Use the following expansion and formula as needed.

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2},$$

$$\int_0^\infty \frac{x^m}{x^n + a^n} dx = \frac{\pi a^{m-n+1}}{n \sin\left(\frac{m+1}{n}\pi\right)} \quad \text{for } 0 < m + 1 < n.$$

DEPARTMENT OF ASTROPHYSICAL SCIENCES,
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

May 12, 2020

9:00 a.m. – 2:00 p.m.

- Today's exam (Part II) contains 6 problems on 2-8 pages. Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed two extra hours, so the total time allotted for today is five hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem on a new page. Put your name and the question number on each page.
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II.1: Wave Quickie [15 points]

Consider magnetized electron–ion plasma with negligible perpendicular temperature. Assume that the magnetic field is parallel to the z axis and consider perturbations with wave vectors $\mathbf{k} = (k_\perp, 0, k_\parallel)$. As you may remember, the dielectric tensor in this case has the same general structure as in cold plasma,

$$\boldsymbol{\epsilon} \approx \begin{pmatrix} \bar{S} & -i\bar{D} & 0 \\ i\bar{D} & \bar{S} & 0 \\ 0 & 0 & \bar{P} \end{pmatrix}. \tag{1}$$

(a) [**3 points**] Assuming the electrostatic approximation, show that the wave dispersion relation in this case can be expressed as follows:

$$k_\perp^2 \bar{S} + k_\parallel^2 \bar{P} \approx 0. \tag{2}$$

(b) [**12 points**] Assume that ions are cold and the wave frequency ω is close enough to the ion cyclotron frequency so that $\bar{S}$ is determined entirely by the ion contribution. Also assume $C_s \ll \omega/k_\parallel \ll v_{Te}$ and $k\lambda_{De} \ll 1$, where C_s is the ion-sound speed, $v_{Te} = \sqrt{T_{\parallel,e}/m_e}$, $T_{\parallel,e}$ is the electron parallel temperature, and λ_{De} is the corresponding Debye length. Based on what you remember from fluid theory, present the corresponding approximations of $\bar{S}$ and $\bar{P}$. Then, show that Eq. (2) leads to

$$\omega^2 \approx \Omega_i^2 + k_\perp^2 C_s^2, \tag{3}$$

which is known as the dispersion relation of electrostatic ion cyclotron waves.

Hint: Note that unlike Eq. (2), Eq. (3) does not contain $k_\parallel$.

II.2: Experimental Quickie [15 points]

Inertial Fusion plasma is confined by transiently compressing the fuel pellet. The radius, R, of the fuel pellet is compressed inward to the center at the sound speed, C_s. The radius during the inward compression is expressed as

$$R(t) = R_0 - C_s t$$

R_0 is the fuel radius at time $t = 0$. The time needed for compression is $\sim T_0 = R_0/C_s$.

(a) [**10 points**] Determine the expression for required confinement time, τ, needed for inertial fusion from the integral, $\tau = \int V(t)/V_0 dt$, where the integral is from $t = 0$ to $t = T_0 = R_0/C_s$. $V(t)$ is the volume of the fuel pellet as a function of time, and V_0 is the initial volume of the fuel pellet at $t = 0$.

(b) [**5 points**] If the sound speed for compression conditions is $C_s = 10^8$cm/s and the fuel radius is 0.5mm, what is the confinement time, τ?

II.3: MHD [45 points]

In this problem, you will be asked to derive minimum energy states under certain assumptions.

(a) [**5 points**] The total electromagnetic energy in a volume, V, is given by

$$W = \int \left(\frac{B^2}{2\mu_0} + \frac{\epsilon_0 E^2}{2} \right) dV, \tag{1}$$

where $\mathbf{B}$ is magnetic field and $\mathbf{E}$ is electric field. Explain why we usually drop the electric field energy in W in MHD. When should we keep it? In the following questions, we will drop the electric field energy in W.

(b) [**5 points**] When a static ideal MHD plasma in V is perturbed by a small velocity $\delta\mathbf{V} \equiv d\xi/dt$ where ξ is the displacement, show that the perturbed magnetic field is given by $\delta\mathbf{B} \equiv \nabla \times (\xi \times \mathbf{B})$.

(c) [**10 points**] Assuming that ξ vanishes on the boundary of V, show that the condition for the perturbed magnetic energy to vanish for any ξ leads to a force-free minimum energy state where $\mathbf{j} \times \mathbf{B} = 0$.

Hint: you can use $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$ *and* $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}\times\mathbf{a}$.

(d) [**15 points**] Derive another force-free minimum energy state by using variational principle with a Lagrange multiplier to minimize W while keeping the total magnetic helicity constant: $K = \int \mathbf{A}\cdot\mathbf{B}dV$, where $\mathbf{A}$ is vector potential of $\mathbf{B}$.

Hint: express the variation of W and K in terms of $\delta\mathbf{A}$ which needs to satisfy the conditions assumed for ξ on the boundary of V.

(e) [**5 points**] What is the difference between these two force-free fields?

(f) [**5 points**] Why are the force-free states not necessarily preferred for plasma confinement?

II.4: Elementary Physics Quickie [15 points]

Statement: X-rays are emitted by plasmas.

(a) [**5 points**] When is this not true or at least at a far lower level of Bremsstrahlung power emitted compared to that in the NRL formulary,

$$P_{Bremss} = 1.69 \times 10^{-32} n_e T_e^{1/2} Z_{\mathrm{eff}} (\mathrm{W/cm^3})?$$

Hint: One answer comes from considering the plasma composition.

(b) [**5 points**] Sketch X-ray spectra, from 0.5 to 2 keV, emitted by

- Pure hydrogen plasma with a $T_e = 0.2$keV Maxwellian electron distribution. Assume the ions are isothermal.
- A nearly identical $T_e = 0.2$keV hydrogen plasma contaminated with 1% oxygen (n_O/n_H). Again, assume the ions are isothermal.

(*Assume the ions' charge state is in coronal equilibrium and the plasma is "transparent", i.e., of appropriately small size and low density.*)

(c) [**5 points**] What are the benefits/drawbacks of a solid-state (pulse-height) detector compared to a grating spectrometer (iCCD-equipped)?

II.5: Applied Math [40 points]

A two-fluid MHD theory on a tokamak resonant layer under a perturbation leads to an equation

$$\frac{dy}{dx} - Fx^2 y = 0 \tag{1}$$

in a simplified geometry, for various parametric regimes such as the resistive-inertial or Hall-resistive regime (A. Cole, 2006). The F depends on many parameters such as electron flow or temperature at the layer. Let F be real and positive for simplicity. The question is to determine the so-called Δ parameter which is the ratio of the two leading terms in small x limit of an exponentially decreasing solution in large x limit. Specifically,

$$\lim_{x\to+0} y(x) \sim y_0 \left(1 - \Delta x + \mathcal{O}(x^2)\right), \quad \text{where} \lim_{x\to+\infty} y(x) \to 0. \tag{2}$$

Here we will solve the Δ analytically as a function of F, with the following steps.

(a) [**8 points**] Find the asymptotic leading behavior of the two linearly independent solutions to y as $x \to +\infty$.

(b) [**12 points**] Obtain an integral representation of the solutions using the Fourier-Laplace Kernel, with a factor representing the dominant exponential behavior in large x limit, i.e. $y = e^{\frac{1}{2}\sqrt{F}x^2} \int_C e^{ixt} f(t)dt$.

(c) [**10 points**] Find a contour path of the integral in (b) that gives the exponentially decreasing characteristics for $x \to +\infty$.

(d) [**10 points**] Expand the contour integral in (c) for small x limit and evaluate the two leading terms in the series to show

$$\Delta = 2F^{1/4}\frac{\Gamma(3/4)}{\Gamma(1/4)}. \tag{3}$$

Hint: Deform the contour path for (c) to the real-axis and split the integral to have a representation with $\int_0^\infty$. *Note that* $\Gamma(z) \equiv \int_0^\infty t^{z-1}e^t dt$ *for* $\mathrm{Re}\,(z) > 0$.

II.6: Waves and Instabilities [50 points]

Here, you are asked to study wave propagation in the Earth's ionosphere. Assume that the ionospheric plasma is collisionless, ions are immobile, and the Earth's curvature is negligible.

(a) [**10 points**] Assume the cold-plasma approximation and neglect the Earth's magnetic field. At some altitude ($\sim$ 300 km), the electron density has the maximum value $n_{\max} = 10^6\ \mathrm{cm}^{-3}$. Suppose an antenna located on the ground and emitting radiation with frequency $f = 12$ MHz at some angle α with respect to the vertical. At what α will this radiation be reflected back to the Earth?

Hint: Reflection occurs when the ***vertical*** *group velocity becomes zero. How does the horizontal wave number, $k_\parallel$, and the frequency evolve along the rays?*

Now, consider the influence of the Earth's magnetic field $\mathbf{B}_0$, assuming it is homogeneous and parallel to the ground. Suppose $\mathbf{k} = (k_\perp, 0, k_\parallel)$, where x is the vertical axis and the z axis is along $\mathbf{B}_0$. The following formulas may be of use, if you know what they mean:

$$S, D = \frac{1}{2}(R \pm L) \qquad R, L = 1 - \frac{\omega_{ps}^2}{\omega\,(\omega \pm \Omega_e)}, \qquad P = 1 - \frac{\omega_p^2}{\omega^2},$$

$$(S\sin^2\theta + P\cos^2\theta)N^4 - [RL\sin^2\theta + PS(1+\cos^2\theta)]N^2 + PRL = 0,$$

$$\tan^2\theta = -\frac{P(N^2-R)(N^2-L)}{(N^2S - RL)(N^2 - P)}.$$

(b) [**10 points**] In magnetized electron plasma, waves can experience reflection at three different locations corresponding to three different values of $X \equiv \omega_p^2/\omega^2$. Find these values $X_{1,2,3}$ as functions of $Y \equiv |\Omega_e|/\omega$ and $N_\parallel \equiv ck_\parallel/\omega \neq 0$. Assuming $0 < Y < 1$ and $N_\parallel^2 \neq 1$, show that there is exactly one value of $N_\parallel^2$, termed $\bar{N}_\parallel^2(Y)$, at which two of the reflection points coincide. Calculate $\bar{N}_\parallel^2(Y)$ and the corresponding value of X.

(c) [**15 points**] Show that $N_\perp \equiv ck_\perp/\omega$ satisfies $\mathsf{a}N_\perp^4 + \mathsf{b}N_\perp^2 + \mathsf{c} = 0$, where a, b, and c depend on X. Find $\mathsf{a}(X)$ and outline how to find $\mathsf{b}(X)$ and $\mathsf{c}(X)$. Without solving this equation, sketch $N_\perp^2(X)$ at fixed $N_\parallel^2$ for $N_\parallel^2 \gtrless \bar{N}_\parallel^2$ and $N_\parallel^2 = \bar{N}_\parallel^2$. You may use

$$\mathsf{b}^2 - 4\mathsf{ac} = \left(\frac{XY}{1-Y^2}\right)^2 \left[Y^2(N_\parallel^2 - 1)^2 - 4(X-1)N_\parallel^2\right].$$

Hint: Consider $N_\parallel = 0$ first, for which case $N_\perp^2(X)$ should be easy to find. (What are the two modes in this regime?) Then, consider how the plot is modified for $N_\parallel \neq 0$ by analyzing $N_\perp^2(0)$, cutoffs, resonance(s), and the number of real roots for $N_\perp^2$.

- Problem continued on next page -

(d) [15 points] Now consider $N_\perp$ in regions where it is real. Sketch $N_\perp(X)$ corresponding to your sketches of $N_\perp^2(X)$ in part (c). (Remember to plot both $N_\perp > 0$ and $N_\perp < 0$.) Using these results, explain the dependence of the field pattern on the launch angle α in the figure below. (Ignore the specific numbers and focus on qualitative physics.)

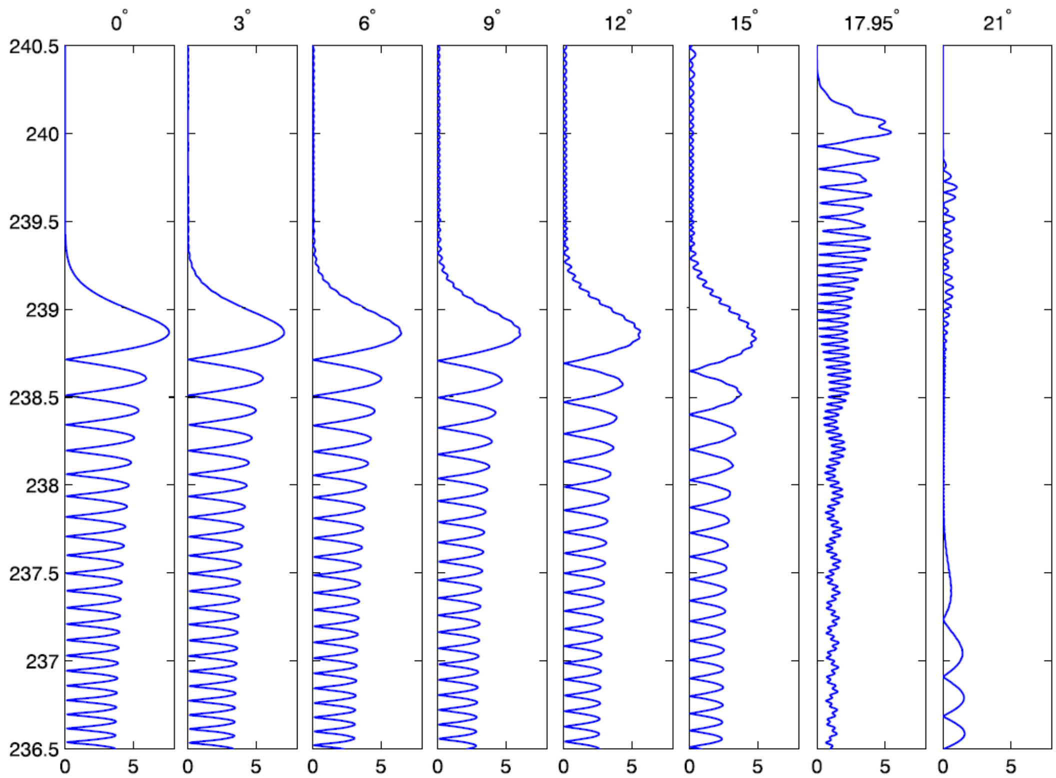

FIG: The absolute value of the wave electric field $|E(x)|$ (horizontal axis) versus the altitude x (vertical axis, in km) for a standing wave launched from the ground ($x = 0$) at different angles α (numbers on top) between $\mathbf{k}$ and the vertical. The figure is adapted from a research paper; the reference is omitted on purpose.