DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

May 3, 2021

9:00 a.m. – 2:00 p.m.

- Today's exam (Part I) contains 6 problems on pages 2–8. Attempt all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed two extra hours, so the total time allotted for today is five hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem on a new page. Put your name and the question number on each page.
- When you do not have time to put answers into forms that satisfy you, indicate ***specifically*** how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem ___" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

Contents

I.1: General Plasma Physics [40 points]

(a) [**6 points**] Derive the trapping condition in midplane ($z = 0$) energy coordinates for ions in a magnetic mirror machine with mirror ratio $R \doteq B_{\max}/B_{\min}$ and mirror axis in the z direction.

(b) [**2 points**] Sketch the trapping condition in the $W_{\perp 0}$–$W_{\parallel 0}$ plane, where $W_{\perp 0}$ ($W_{\parallel 0}$) is the perpendicular (parallel) energy as the midplane is crossed.

(c) [**4 points**] Suppose that the magnetic field near the axis can be approximated as

$$\boldsymbol{B} = \begin{cases} B_0\,(1 + z^2/L^2)\,\hat{\boldsymbol{z}} & \text{if } z^2/L^2 < c^2, \\ B_0\,(1 + c^2)\,\hat{\boldsymbol{z}} & \text{if } z^2/L^2 \geq c^2, \end{cases}$$

where c is a constant. Show that the turning points $\pm z_{\mathrm{T}}$ for trapped ions obey $z_{\mathrm{T}}^2/L^2 = W_{\parallel 0}/W_{\perp 0}$.

Suppose now that the mirroring particles are in an axial gravitational field, $-g\hat{\boldsymbol{z}}$.

(d) [**4 points**] Write and sketch the new trapping condition in midplane ($z = 0$) energy coordinates.

(e) [**4 points**] Derive the new turning points, defining the high turning point as z_{H} and the low turning point as z_{L}. Express z_{H} and z_{L} in terms of z_{T} and z_g, where $z_g \doteq mgL^2/W_{\perp 0}$. Show that trapped particles always cross the midplane.

Suppose now that the axial gravitational field is not constant, but instead increases very slowly in time from zero at time $t = 0$ to a finite value g at time $t = t_{\mathrm{f}}$. (Hint: It might be convenient to write the parallel energy as a function of $W_{\perp 0}$, z_{H}, z_{L} and z.)

(f) [**2 points**] What is the change in the perpendicular midplane energy $W_{\perp 0}$?

(g) [**8 points**] What is the change in the parallel midplane energy $W_{\parallel 0}$? You might find the following integral useful:

$$\int_a^b \mathrm{d}s\,\left[(s-a)(b-s)\right]^{1/2} = \frac{\pi}{8}(b-a)^2.$$

(h) [**6 points**] Write down the condition (in terms of the initial, $t = 0$, $W_{\perp 0}$ and $W_{\parallel 0}$ coordinates) for particles that are initially trapped but then detrapped. If there is a region in $W_{\perp 0}$–$W_{\parallel 0}$ space containing any such particles, what is the minimum $W_{\perp 0}$ and the minimum $W_{\parallel 0}$ bounding this region?

(i) [**4 points**] Suppose instead that the axial gravitational field starts off at value g at time $t = 0$, and then decreases very slowly in time to zero at time $t = t_{\mathrm{f}}$. Write down the condition (in terms of the initial $W_{\perp 0}$ and $W_{\parallel 0}$) for particles that are initially trapped but then detrapped. If there is a region in $W_{\perp 0}$–$W_{\parallel 0}$ space containing any such particles, what is the minimum $W_{\perp 0}$ and the minimum $W_{\parallel 0}$ bounding this region?

I.2: Waves Short Problem [15 points]

Consider a collisionless one-dimensional electron plasma with stationary ions. Assume that the background electron distribution has the form $f_0(v) = F(v^2)$ with $F'(\varepsilon) < 0$ for all ε.

(a) [**7 points**] Assume a perturbation to the electron distribution of the form $\widetilde{f} \propto \exp(-\mathrm{i}\omega t + \mathrm{i}kx)$. Using the linearized Vlasov equation and *Ampere's* law, derive the dispersion relation for electrostatic oscillations of this form.

(b) [**8 points**] Discretize the velocity space into N intervals of length Δv, each centered about $v = v_a$ with $a = 1, 2, \ldots, N$. Show that the dispersion relation derived in part (a) can be written as

$$\omega = \sum_{a=1}^{N} \frac{g_a^2}{\omega - kv_a} \quad \text{with} \quad g_a^2 > 0.$$

Argue that all solutions for $\omega(k)$ are real. Where is Landau damping in this model?

I.3: MHD Short Problem [15 points]

A small-amplitude, linearly polarized Alfvén wave of amplitude $B_\perp$ and wavenumber $k > 0$ propagates along a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ through an otherwise stationary, uniform, ideal-MHD plasma. The magnetic field and fluid velocity are given by

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + B_\perp \sin[k(z - v_{\rm A}t)]\,\hat{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{u} = -v_{\rm A}\frac{B_\perp}{B_0}\sin[k(z - v_{\rm A}t)]\,\hat{\boldsymbol{x}},$$

respectively, where $v_{\rm A} \doteq B_0/\sqrt{4\pi\rho}$ is the Alfvén speed.

(a) [**9 points**] Neglecting terms of order $B_\perp^2$ and higher, compute the current density $\boldsymbol{j}_{\rm pol}$ associated with the particles' polarization drift in this wave and show that it is equal to the total current density from Ampère's law, $\boldsymbol{j} = (c/4\pi)(\boldsymbol{\nabla}\times\boldsymbol{B})$.

(b) [**6 points**] There is also a current associated with the curvature drift, equal to

$$\boldsymbol{j}_{\rm curv} = \frac{ckB_\perp}{4\pi}\,\frac{4\pi p}{B_0^2}\cos[k(z - v_{\rm A}t)]\,\hat{\boldsymbol{y}},$$

where p is the thermal pressure. If $\boldsymbol{j}_{\rm pol} = \boldsymbol{j}$, then what balances $\boldsymbol{j}_{\rm curv}$? Prove it.

I.4: Irreversible Processes [55 points]

This problem revisits the Spitzer–Härm problem of calculating the electrical conductivity σ of a collisional, electron–ion plasma, only this time it's for a magnetized plasma in which the conductivity is a tensor, $\boldsymbol{\sigma}$. Mathematical formulae of possible utility are provided at the end of the problem. *Attempt all parts! Partial credit may be earned!* For example, note that you can answer parts (d)–(f) without even attempting (a)–(c).

A constant, uniform electric field $\boldsymbol{E} = E_\parallel \hat{\boldsymbol{b}} + \boldsymbol{E}_\perp$ is applied to a quasi-neutral, electron–ion plasma threaded by a constant, uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{b}}$. In steady state, the kinetic equation describing the electron distribution function $f_e = f_e(\boldsymbol{v})$ then reads

$$-\left(\frac{eE_\parallel}{m_e}\hat{\boldsymbol{b}} + \frac{e\boldsymbol{E}_\perp}{m_e} + \Omega_e \boldsymbol{v}\times\hat{\boldsymbol{b}}\right)\boldsymbol{\cdot}\frac{\partial f_e}{\partial \boldsymbol{v}} = C[f_e], \tag{1}$$

where $\Omega_e \doteq eB/m_e c$. The notation is standard: e is the electric charge, m_e is the electron mass, c is the speed of light, and $\boldsymbol{v}$ is the velocity-space coordinate. In what follows, neglect any ion motion and take the collision operator on the right-hand side of equation (1) to be the Lorentz operator describing electron–ion collisions:

$$C[f_e] = \frac{\nu(v)}{2}\left[\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial^2}{\partial\phi^2}\right] f_e \doteq \nu(v)\mathcal{L}[f_e] \quad \text{with} \quad \nu(v) \doteq \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\text{the}}}{v}\right)^3, \tag{2}$$

where $v_{\text{the}} \doteq (2T_e/m_e)^{1/2}$ is the electron thermal speed for temperature T_e, τ_{ei} is the electron–ion collision timescale, and $\boldsymbol{v} = v_\parallel \hat{\boldsymbol{b}} + \boldsymbol{v}_\perp = v\xi\,\hat{\boldsymbol{b}} + v\sqrt{1-\xi^2}\,(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}})$ with $\hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}} = \hat{\boldsymbol{b}}$.

(a) [**8 points**] Write down the definition of the Dreicer field E_{D}. Then order the dimensionless parameters that appear in equation (1) as follows:

$$\frac{E_\perp}{E_{\text{D}}} \sim \frac{E_\parallel}{E_{\text{D}}} \doteq \epsilon \ll 1, \quad \Omega_e\tau_{ei} \sim 1.$$

Expand the electron distribution function in powers of ϵ as $f_e = f_{e0} + \epsilon f_{e1} + \epsilon^2 f_{e2} + \dots$ and write out equation (1) at $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ relative to $\Omega_e f_{e0}$. Use your $\mathcal{O}(1)$ equation to prove that f_{e0} is isotropic in velocity space, i.e., $f_{e0} = f_{e0}(v)$.

(b) [**21 points**] Solve your $\mathcal{O}(\epsilon)$ equation for f_{e1}. In doing so, follow Braginskii and conduct a subsidiary expansion in $\delta \doteq (\Omega_e\tau_{ei})^{-1} \ll 1$ to determine f_{e1} out to $\mathcal{O}(\epsilon\delta^2 f_{e0})$.

Hints: Split f_{e1} into a gyro-averaged part, $\langle f_{e1}\rangle_\phi$, and a gyrophase-dependent part, $\widetilde{f}_{e1}$. You may also find it simpler to orient $\hat{\boldsymbol{b}} = \hat{\boldsymbol{z}}$ and $\boldsymbol{E}_\perp = E_\perp\hat{\boldsymbol{x}}$ without loss of generality. Because f_{e0} is isotropic (see part a), your answer should be proportional to $\mathrm{d}f_{e0}/\mathrm{d}v$.

(Problem continues on next page.)

(c) [**12 points**] Now take f_{e0} to be Maxwellian,

$$f_{e0}(v) = \frac{n_e}{\pi^{3/2} v_{\mathrm{the}}^3} \exp\left(-\frac{v^2}{v_{\mathrm{the}}^2}\right),$$

and use your answer from part (b) to show that the parallel current density is given by

$$\boldsymbol{j}_\parallel = \frac{32}{3\pi} \sigma E_\parallel \hat{\boldsymbol{b}}, \tag{3}$$

where $\sigma \doteq e^2 n_e \tau_{ei}/m_e$.

If you could not do part (b), then state how you would compute $\boldsymbol{j}_\parallel$ *given* f_{e1} *and explain which part of* f_{e1} *contributes to* $\boldsymbol{j}_\parallel$ *and why.*

(d) [**4 points**] If you were to use the full Landau collision operator for $C[f_e]$ instead of just the Lorentz operator given by equation (2), would the resulting coefficient on $j_\parallel$ in equation (3) be larger or smaller than $32/3\pi \simeq 3.40$? Briefly explain your answer.

(e) [**4 points**] The complete expression for the current density $\boldsymbol{j}$ obtained from f_{e1} is

$$\boldsymbol{j} \doteq \boldsymbol{j}_\parallel + \boldsymbol{j}_\times + \boldsymbol{j}_\perp = \frac{32}{3\pi} \sigma E_\parallel \hat{\boldsymbol{b}} - \frac{\sigma}{\Omega_e \tau_{ei}} \boldsymbol{E} \times \hat{\boldsymbol{b}} + \frac{\sigma}{(\Omega_e \tau_{ei})^2} \boldsymbol{E}_\perp \doteq \boldsymbol{\sigma} \boldsymbol{\cdot} \boldsymbol{E}. \tag{4}$$

In words, the electrical conductivity of a magnetized plasma is a tensor that is biased with respect to the magnetic-field direction. Use random-walk arguments to explain physically why $j_\perp \sim j_\parallel (\Omega_e \tau_{ei})^{-2} \ll j_\parallel$.

(f) [**6 points**] The "$\times$" conductivity, *viz.* $\sigma_\times \doteq \sigma/(\Omega_e \tau_{ei}) = e n_e c/B$, is independent of collisions. In a few sentences or using a diagram, explain what physics causes $\sigma_\times$.

Possibly useless information:

$$\text{for } (v_\parallel, v_\perp, \phi) \text{ coordinates,} \quad \frac{\partial}{\partial \boldsymbol{v}} = \hat{\boldsymbol{b}} \frac{\partial}{\partial v_\parallel}\bigg|_{v_\perp, \phi} + \frac{\boldsymbol{v}_\perp}{v_\perp} \frac{\partial}{\partial v_\perp}\bigg|_{v_\parallel, \phi} - \frac{\boldsymbol{v} \times \hat{\boldsymbol{b}}}{v_\perp^2} \frac{\partial}{\partial \phi}\bigg|_{v_\parallel, v_\perp}$$

$$\langle \boldsymbol{v} \rangle_\phi = v_\parallel \hat{\boldsymbol{b}}, \quad \langle \boldsymbol{v}\boldsymbol{v} \rangle_\phi = v_\parallel^2 \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \frac{v_\perp^2}{2} (\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})$$

$$\frac{\partial \boldsymbol{v}_\perp}{\partial \phi} = -\boldsymbol{v} \times \hat{\boldsymbol{b}}, \qquad \frac{\partial (\boldsymbol{v} \times \hat{\boldsymbol{b}})}{\partial \phi} = \boldsymbol{v}_\perp, \qquad \frac{\partial \boldsymbol{v}_\perp}{\partial \xi} = -\frac{\xi}{1 - \xi^2} \boldsymbol{v}_\perp$$

$$\boldsymbol{A} \boldsymbol{\cdot} (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \boldsymbol{\cdot} (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \boldsymbol{\cdot} (\boldsymbol{A} \times \boldsymbol{B}), \qquad \boldsymbol{A} \boldsymbol{\cdot} (\boldsymbol{B} \times \mathsf{I}) = \boldsymbol{A} \times \boldsymbol{B}$$

$$\int_0^\infty \mathrm{d}x\, x^k\, \mathrm{e}^{-x} \doteq \Gamma(k+1) \ \big(= k! \text{ for integer } k \geq 0\big)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

$$\mathcal{L}[P_\ell(\xi)] = -\frac{\ell(\ell+1)}{2} P_\ell(\xi), \quad P_0(\xi) = 1, \quad P_1(\xi) = \xi, \quad P_2(\xi) = \frac{1}{2}(3\xi^2 - 1)$$

I.5: Applied Math Short Problem [15 points]

Consider a harmonic oscillator where the frequency $\Omega(t)$ is changing slowly:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \Omega^2(t) y = 0. \tag{1}$$

(a) [**10 points**] Find a leading-order asymptotic solution appropriate to the slowly changing frequency. How slow should it be in order to justify your solution?

(b) [**5 points**] Show that the action integral of this nearly periodic motion,

$$\mathcal{J} = \oint \mathrm{d}y \, \frac{\mathrm{d}y}{\mathrm{d}t}, \tag{2}$$

is proportional to E/Ω, where E is the total energy of the system.

I.6: Experimental Methods [40 points]

The following questions on experimental methods are grouped into two parts. Attempt both parts.

Part I: Probes. A floating planar probe is immersed into a weakly collisional, steady-state, low-temperature plasma whose electrons have a Maxwellian energy distribution with temperature T_e. Take the ions to be cold, with temperature $T_i \ll T_e$.

(a) [**20 points**] The probe is heated to temperatures at which it starts to emit electrons. Assume that the temperature of the emitted electrons is negligible compared to the temperature of the plasma electrons. Derive an expression for the floating potential of the electron emitting probe with respect to the sheath–pre-sheath edge.

(b) [**5 points**] A sweeping bias voltage is applied to the probe. The bias voltage is swept to get the full probe V–I characteristic. Provide a qualitative sketch of this characteristic at different probe temperatures (and therefore different electron emission) to show how the probe emission affects the V–I curve.

Part II: Thomson scattering.

(c) [**10 points**] Derive the Thomson scattering cross-section in the non-collective regime. You are given the Larmor formula for power radiated by an accelerated charge,

$$P_{\text{rad}} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c^3}\dot{v}^2.$$

(d) [**5 points**] Calculate the fraction of photons incoherently scattered from a 1 cm path length of laser beam off a plasma with an electron density of 2×10^{20} m^{-3} into a solid angle of detection of 0.01 sr. Useful constant: the classical electron radius $r_{\text{e}} \simeq 2.8 \times 10^{-15}$ m.

DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

May 4, 2021

9:00 a.m. – 2:00 p.m.

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- Today's exam has been designed to require three hours of work (180 points). However, you are allowed two extra hours, so the total time allotted for today is five hours. The scores on the questions will be weighted in proportion to their allotted time.
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II.1: General Plasma Physics [30 points]

Consider the electrostatic dynamics in a plasma governed (strictly) by the Vlasov–Poisson equations

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_s - \frac{q_s}{m_s} \boldsymbol{\nabla} \varphi \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = 0, \tag{1}$$

$$\nabla^2 \varphi = -4\pi \sum_s q_s \int \mathrm{d}^3\boldsymbol{v}\, f_s. \tag{2}$$

(a) [**1 point**] The total energy E of the system is defined as

$$E \doteq \int \mathrm{d}^3\boldsymbol{r}\, \mathscr{E},$$

$$\mathscr{E} \doteq \sum_s \int \mathrm{d}^3\boldsymbol{v}\, \frac{1}{2} m_s v^2 f_s + \frac{1}{8\pi} \left| \boldsymbol{\nabla} \varphi \right|^2.$$

What is the physical meaning of $\mathscr{E}$?

(b) [**5 points**] Show that the total energy of the system is conserved, i.e.,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0.$$

(c) [**2 points**] The Vlasov–Poisson system (1)–(2) admits an exact local energy conservation law in the form of

$$\frac{\partial \mathscr{E}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathscr{F} + \mathscr{P}) = 0,$$

where

$$\mathscr{F} \doteq \sum_s \int \mathrm{d}^3\boldsymbol{v}\, \frac{1}{2} m_s v^2 \boldsymbol{v} f_s.$$

What is the physical meaning of $\mathscr{F}$?

(d) [**22 points**] Derive an expression for $\mathscr{P}$. What is its physical meaning?

II.2: MHD [30 points]

This problem concerns the "Parker spiral" model describing the geometry of the magnetic field emanating from the Sun and filling the interplanetary medium. Assume that the solar wind's magnetic field is "frozen" in the plasma and obey's Alfvén's theorem.

(a) [**10 points**] Assume at first that the Sun is an ideal sphere, does not rotate, and that the solar wind expands *radially* out from the surface of the Sun at a constant speed V_{sw}. Obtain an expression for the radial component of the magnetic field, assuming that its magnitude on the Sun's surface is B_{r0}.

(b) [**10 points**] Now let the Sun rotate with an angular speed ω_{S} about its polar axis. The initially radial magnetic field will then be stretched out in a spiral (known as the Parker spiral in heliophysics). Obtain an equation for this spiral geometry in the equatorial plane.

(c) [**10 points**] Obtain expressions for the radial and azimuthal components of the magnetic field in the equatorial plane. Which component dominates at large radial distances?

II.3: Waves and Instabilities [50 points]

Consider a cold stationary electron–ion plasma with one type of ions and a background magnetic field $\boldsymbol{B}_0$ along the z axis. The plasma is homogeneous along the y and z axes, but the background density and B_0 depend on x. In this plasma, consider stationary waves with electric field $\widetilde{\boldsymbol{E}} = \boldsymbol{E}(x)\,\mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}k_z z}$ with $k_z > 0$ and $0 < \omega/\Omega_i \ll 1$. Use that

$$\boldsymbol{\epsilon} = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}, \qquad S \approx \frac{\omega_{\mathrm{p}i}^2}{\Omega_i} \gg 1, \qquad D \approx -\frac{\omega}{\Omega_i}\,S.$$

(a) [**3 points**] At $\omega \to 0$, $\widetilde{\boldsymbol{E}}$ becomes stationary, and one might expect that a stationary field cannot create a current perpendicular to $\boldsymbol{B}_0$. Then, why is $S - 1$ nonzero?

(b) [**10 points**] Assume that P is large enough such that E_z is negligible. Starting from Maxwell's equations and assuming the notation $N_z \doteq ck_z/\omega$, show that E_y satisfies

$$\left(\frac{c^2}{\omega^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + S - N_z^2 - \frac{D^2}{S - N_z^2}\right) E_y = 0. \tag{1}$$

(c) [**12 points**] Assume $S = (1 + x/L)N_z^2$, so that $x = 0$ corresponds to the so-called Alfvén resonance, where $S = N_z^2$. Assume $L > 0$. Argue that Eq. (1) can be written as

$$\left(x\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2 - 1}{2\alpha^2}\right) E_y = 0, \tag{2}$$

where x has been appropriately rescaled and α is a constant that you are asked to find. Plot the geometrical-optics (GO) dispersion curves $k_x(x) \lesseqgtr 0$ corresponding to Eq. (2). Also find and plot the inverse function, which has the form $x(k_x) = \bar{x}(k_x) \pm \Delta(k_x)$.

(d) [**12 points**] In the GO limit, calculate the x-component of the group velocity at $S - N_z^2 \gg D$ (for $x > 1$) and $S - N_z^2 \ll D$ (for $-1 < x < 0$) to determine the direction of the action flows along the dispersion curves. Identify the branches at $x \to +\infty$ and at $x \to 0-$. Describe what happens to a wave launched toward the Alfvén resonance.

(e) [**13 points**] Consider $g_1(k_x) \doteq \mathrm{e}^{\mathrm{i}\Theta(k_x)} \int \mathrm{d}x\, E_y(x)\,\mathrm{e}^{\mathrm{i}k_x x}$, where $\Theta(k_x) \doteq \int \mathrm{d}k_x\, \bar{x}(k_x)$ and $\bar{x}$ is the same as in part (c). Take for granted that g_1 and $g_2 \doteq -\mathrm{i}sg_1 - g_1'$ satisfy

$$\mathrm{i}\,\Delta k_x \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} s & -\mathrm{i} \\ \mathrm{i} & -s \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \qquad s \doteq \alpha^2 k_x^2. \tag{3}$$

Assume that $g_{1,2} \propto \mathrm{e}^{-\mathrm{i}\theta(k_x)}$, where $\zeta \doteq \theta'(k_x)$ changes slowly with k_x compared with θ. Derive the two corresponding GO dispersion branches $\zeta_{1,2}(k_x)$ and explain how they are related to $x(k_x)$ in part (c) *and why*. By considering the "frequency gap" $|\zeta_1 - \zeta_2|$, estimate the parameter that determines the mode-conversion efficiency. Is energy deposition at the Alfvén resonance larger at small α or at large α?

II.4: Toroidal Magnetically Confined Plasmas [23 points]

Answer the following questions on toroidal magnetically confined fusion plasmas.

(a) [**13 points**] In a toroidal magnetically confined fusion plasma, what are the safety factor, q, and the rotational transform, ι? Give an expression for the safety factor in a large aspect ratio tokamak with circular cross-section, and then write it in terms of the toroidal current in the tokamak. Is there a similar expression for the rotational transform in a large aspect ratio stellarator in terms of the current? in terms of the toroidal and poloidal fields? Explain your answer.

(b) [**4 points**] Name two instabilities that cause turbulent transport in tokamaks.

(c) [**6 points**] Name three types of RF heating that are used in tokamaks.

II.5: Applied Math [35 points]

In the theory of mode conversion of some plasma waves we arrive at the following integral:

$$A(x) \doteq \frac{1}{\mathrm{i}} \int_0^\infty \mathrm{d}k \, \exp\left(-\mathrm{i}kx + \frac{\mathrm{i}}{5}k^5\right).$$

(a) [**5 points**] Show that $A(x)$ is a solution of the inhomogeneous Hyper-Airy equation

$$\frac{\mathrm{d}^4 A}{\mathrm{d}x^4} - xA = 1.$$

(b) [**5 points**] Evaluate the first two terms of the power series expansion of $A(x)$ for $x \ll 1$.

(c) [**5 points**] Find the saddle points of the integral for $x \gg 1$ and $-x \gg 1$.

(d) [**8 points**] Sketch the steepest descent paths for $x \gg 1$ and $-x \gg 1$.

(e) [**12 points**] Calculate the leading order asymptotic value for both the real **and** the imaginary parts of $A(x)$ for $-x \gg 1$.

You may need to express some answers in terms of the gamma function, defined by

$$\Gamma(y) \doteq \int_0^\infty \mathrm{d}s \, s^{y-1} \mathrm{e}^{-s}.$$

II.6: Experimental Short Problem [12 points]

For **each** of the plasmas whose characteristics are provided in the table below, describe **two** ways that are commonly used to measure the electron density.

	n_e (cm^{-3})	T_e (eV)	B (G)	n_n (cm^{-3})	plasma type
1	10^9	10	0	10^{14}	DC glow
2	10^{13}	10^3	10^4	10^{12}	magnetic fusion
3	10^{24}	10^3	10^6	0	inertial confinement

Here, n_e is the electron number density, T_e is the electron temperature, B is the magnetic field, and n_n is the neutral number density (if necessary, assume atomic hydrogen).