DEPARTMENT OF ASTROPHYSICAL SCIENCES

PLASMA PHYSICS SECTION

General Examination

Part I (May 16, 1988)

- ***Answer all problems!***
- The total time allotted for this part is ***3 1/2 hours***. Scores on questions will be weighted in proportion to the allotted time.
- Start each numbered problem on a *new page*. Put your name on every page.
- When you do not have time to put answers into forms that satisfy you, indicate *specifically* how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem __." and sign your name.
- *Notation:* We attempt to be consistent and use the construction $a \stackrel{\text{def}}{=} expression$ to mean "a is defined to be *expression*." Many people use '$\equiv$' instead of '$\stackrel{\text{def}}{=}$'.

Good luck!!!

PROBLEM 1: *General physics* [20 minutes total].

The intrepid space explorer Snake Simon lands on the isothermal planet Ferth. Blowing on his 30 cm open-ended reed pipe, he finds that it sounds middle A (440 hz), and dangling his Girard–Perregaux pocket watch from its 40 cm gold chain, he times 10 full oscillations in 15 seconds. What is the scale height of the Ferthian atmosphere?

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PROBLEM 2: *Kinetic theory* [15 minutes total].

Consider a gas of N diatomic molecules immersed in a *uniform* electric field. Suppose that each molecule can be modeled as two equal masses of opposite charges connected by a rigid rod of length L.

[10 min.] **(2a)** Write down a Liouville equation for the system of molecules, neglecting any fields generated by the molecules themselves (*i.e.*, assume that the molecules are noninteracting). What is the dimension of the space in which the distribution function evolves?

[5 min.] **(2b)** List the important properties of the equation that you have written down. Name and express mathematically these properties. [***Hint:*** You may be able to surmise the answer to part (b) even if you do not complete part (a).]

* * *

PROBLEM 3: *Transport* [30 minutes total].

[5 min.] **(3a)** In estimating classical particle diffusion for electrons and hydrogen ions in a uniform magnetic field (*i.e.*, cylindrical geometry with $\vec{B} = B_0\hat{z}$), the spatial change in the guiding-center position of each particle is given by $\Delta\vec{r} = \Delta\vec{v}\times\hat{z}/\omega_c$ (with ω_c being the gyrofrequency and $\Delta\vec{v}$ repesenting the change in the velocity due to collisions). Use this expression to illustrate the influence of like-particle and unlike-particle collisions on diffusion.

[10 min.] **(3b)** For a toroidal geometry a fluid-type analysis is often used to calculate the so-called Pfirsch–Schlüter diffusion coefficient. Here, however, show how to use heuristic "random-walk" arguments to estimate D_{PS}. How does it compare to the ordinary classical diffusion coefficient D_{cl} in the simple system described in part (a)? Describe the collisionality regime in which such estimates are appropriate.

[15 min.] **(3c)** Describe the so-called banana regime for a tokamak and give an estimate for the diffusion coefficient in this regime. Justify your choice for the radial step size by invoking conservation of canonical angular momentum.

* * *

PROBLEM 4: *Experimental* [15 minutes total].

A plasma is created by an rf discharge whose frequency is much less than the plasma frequency. Assume that the plasma parameters (*e.g.*, temperature or density) do not change in time. The plasma potential can be measured by a floating probe. The plasma potential oscillates around the ground potential (*i.e.*, $V = V_0 \sin \omega t$).

You are asked to obtain probe characteristics by using another Langmuir probe. How would you do it? Give a rough sketch of the arrangement.

Hint: A resistor, an oscilloscope, and wires *etc.* should be sufficient.

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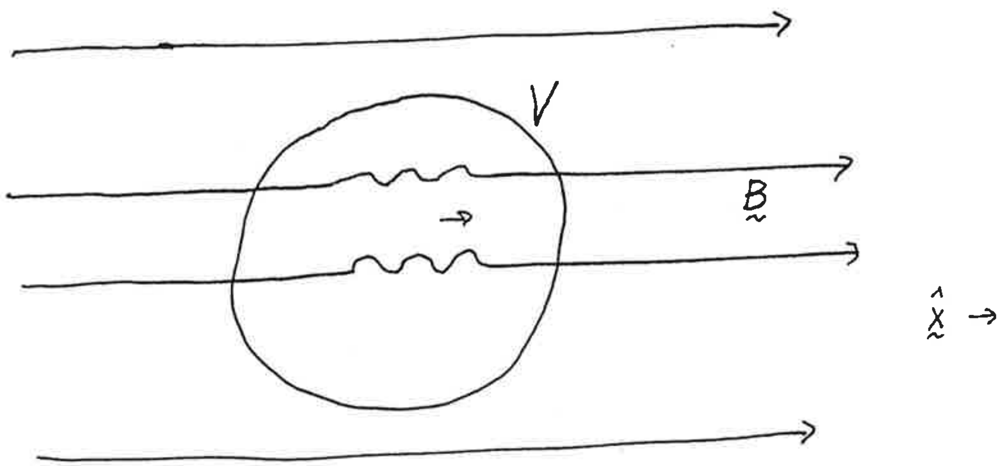

Fig. 1: A disturbance localized to volume V.

PROBLEM 5: *MHD* [35 minutes total].

Consider an infinite homogeneous plasma in a homogeneous magnetic field in the x direction:

$$\vec{B} = B_0 \hat{x}.$$

Let there be a localized disturbance in a volume V with the plasma undisturbed outside of V. (See Fig. 1.)

[15 min.] **(5a)** Show from the ideal MHD equations that the total momentum $\vec{P}_{\text{wave}}$,

$$\vec{P}_{\text{wave}} \stackrel{\text{def}}{=} \int_V d\vec{x}\, \rho \vec{v},$$

is constant in time.

[20 min.] **(5b)** Let the disturbance be a plane-polarized *shear*-Alfvén wave packet propagating in the positive x direction (along $\vec{B}$) with velocity c_A. Let the gas pressure p vanish. By working to second order, calculate the momentum P_{wave} and energy E_{wave} of the wave and show that

$$\frac{P_{\text{wave}}}{E_{\text{wave}}} = \frac{1}{2c_A}$$

(not $1/c_{\text{A}}$*!).* ***Hint:*** You may assume that the momentum is in the direction of wave propagation.

* * *

PROBLEM 6: *Experimental* [15 minutes total].

The global energy confinement time τ_E is one of the most commonly used experimental quantities for gauging the quality of a tokamak plasma. In a steady-state situation, τ_E can be defined to be the ratio of total plasma energy to input power needed to sustain the plasma. In estimating the total plasma energy, there are two main experimental approaches—namely, the magnetic and the kinetic approaches. Describe briefly what these are and list diagnostics used. What are their advantages and disadvantages?

* * *

PROBLEM 7: *Quasilinear theory I* [10 minutes total].

[4 min.] **(7a)** How does the derivation of quasilinear theory avoid dealing with the occurrence of particles trapped in the moving waves?

[6 min.] **(7b)** A. N. Kaufman divides the diffusion function into a part that yields a component of $f(\vec{v},t)$ whose *amplitude* is proportional to $|E|^2$ and a part that yields a component whose *rate of evolution* is proportional to $|E|^2$. What are applications for each of these components? List as many as you can.

* * *

PROBLEM 8: *Quasilinear theory II* [40 minutes total].

In this problem, we shall employ quasilinear theory and consider the effects of ICRF heating on ions in large-aspect-ratio tokamaks with circular magnetic surfaces. Referring to Fig. 2, we have $\omega \approx \omega_c(R_c)$ (*i.e.*, $\omega = eI_0/McR_c$); R_b corresponds to the banana tips of the trapped ions (ignore the finite banana widths here).

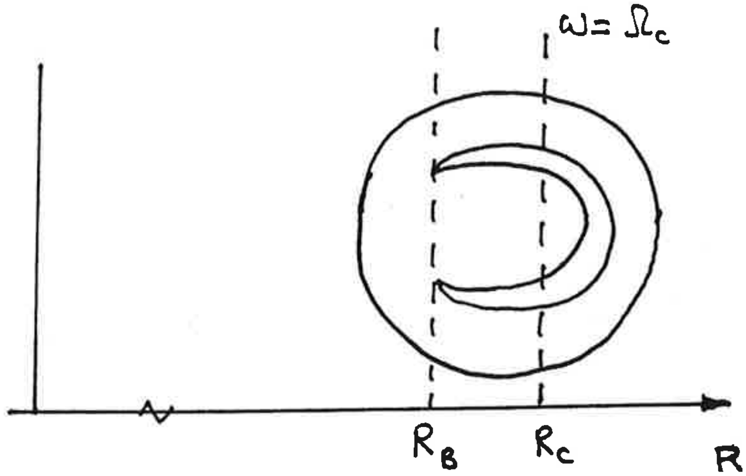

Fig. 2: Minor cross-section of a tokamak, showing the nominal resonance radius R_c and the radius R_b of the banana tips of the trapped ions.

Since the time scale of quasilinear diffusion is much longer than the bounce or transit periods of the ions, we have, in the lowest-order approximation, the following quasilinear equation for the bounce/transit-averaged distribution function F_0:

$$\frac{\partial}{\partial t} F_0(\epsilon, \lambda) = \mathcal{L} \mathcal{D} \mathcal{L} F_0, \tag{8.1}$$

where $\epsilon \overset{\text{def}}{=} v^2/2$, $\lambda \overset{\text{def}}{=} \mu/\epsilon$ (where $\mu \overset{\text{def}}{=} v_\perp^2/2B$) defines the pitch angle,

$$D \overset{\text{def}}{=} g(\epsilon, \lambda) |\delta \vec{E}|^2 > 0, \tag{8.2}$$

$$g(\epsilon{=}0, \lambda) = g(\epsilon, \lambda{=}0) = g(\epsilon, \lambda{=}1/B_{\min}) = 0, \tag{8.3}$$

$B_{\min}$ is the minimum value of B on a given flux surface, and

$$\mathcal{L} \stackrel{\text{def}}{=} \frac{\partial}{\partial \epsilon} + \left(\frac{\omega_c}{B\omega} - \lambda\right) \frac{1}{\epsilon} \frac{\partial}{\partial \lambda}. \tag{8.4}$$

(We have suppressed the dependence on the minor radius.)

[25 min.] **(8a)** Show that time-asymptotically F_0 exhibits the quasilinear "flattening" along a certain characteristic $K = K(\epsilon, \lambda)$. That is, prove that

$$\lim_{t \to \infty} F_0 \to \overline{F}_0(K), \tag{8.5}$$

where

$$K \stackrel{\text{def}}{=} \epsilon(1 - B\omega\lambda/\omega_c). \tag{8.6}$$

Hint: Calculate the time-asymptotic evolution of the velocity-space average of F_0^2. Note the following useful relation of magnetic flux-tube averaging of velocity-space integrations:

$$\oint \frac{d\ell}{B} \int d\vec{v}\, F_0^2 = \tau_b \int_0^\infty \epsilon\, d\epsilon \int_0^{1/B_{\min}} d\lambda\, F_0^2,$$

where

$$\tau_b = \oint \frac{d\ell}{|v_{\|}|}.$$

[15 min.] **(8b)** Prove that for very energetic ($\epsilon \to \infty$) ions produced by the ICRF heating, the asymptotic distribution function $\overline{F}_0$, Eq. (8.5), dictates that all ions are trapped and their banana tips at $R = R_b$ coincide with the resonance layer at $R = R_c$. (See Fig. 2.) In other words, as the trapped ions are heated by the ICRF waves, their banana tips become time-asymptotically localized around the resonance layer. This effect is called "resonance localization."

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PROBLEM 9: *Applied math* [30 minutes total].

Trapped particles can destabilize an MHD mode. A model dispersion relation for the frequency ω is

$$0 = -A - i\frac{\omega}{\omega_{\rm A}} + \beta\frac{\omega}{\omega_{\rm d}} \ln\left(1 - \frac{\omega_{\rm d}}{\omega}\right),$$

where β is the trapped-particle beta, $\omega_{\rm d}$ is the trapped-particle precession rate, $\omega_{\rm A}$ is the Alfvén frequency, and A is a *real* number depending on the equilibrium parameters.

[10 min.] **(9a)** For $0 < \omega < \omega_{\rm d}$, find approximate bounds on A such that there exist one, two, or no thresholds.

[10 min.] **(9b)** Find the values of β associated with each threshold frequency.

[10 min.] **(9c)** For the case $A = 0$, expand the $\omega \neq 0$ branch around threshold to find the dependence of the growth rate on β.

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DEPARTMENT OF ASTROPHYSICAL SCIENCES

PLASMA PHYSICS SECTION

General Examination

Part II (May 17, 1988)

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- The total time allotted for this part is ***3 1/2 hours.*** Scores on questions will be weighted in proportion to the allotted time.
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Good luck!!!

PROBLEM 1: *Dimensionless parameters* [15 minutes total].

What dimensionless number or numbers characterize, by being large or small compared to unity,

[3 min.] **(1b)** The plasma state?

[4 min.] **(1a)** Adiabaticity of a particle orbit in a magnetic field?

[4 min.] **(1c)** "Hot" *vs.* "cold" plasma wave analysis?

[4 min.] **(1d)** The effect of resistivity on magnetohydrodynamic motions?

In each case, justify your answer very briefly.

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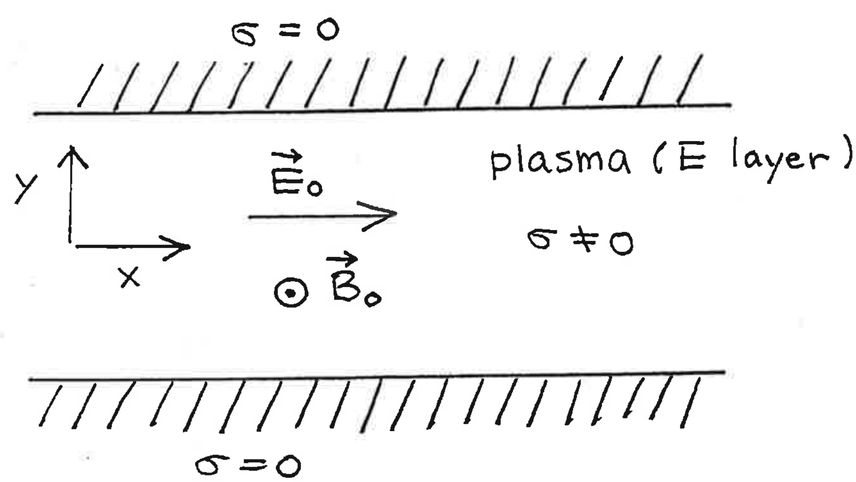

Fig. 3: Geometry of the E layer.

PROBLEM 2: *Two-fluid equations* [30 minutes total].

Consider a weakly ionized cold plasma immersed in a uniform external magnetic field $\vec{B} = B_0\hat{z}$ and an electric field $\vec{E} = E_0\hat{x}$ (such as in the ionosphere), where E_0 is a constant. Consider the two-fluid model for the electrons and ions in steady state:

$$0 = -\frac{e}{m_e}(\vec{E} + \frac{1}{c}\vec{v}_e \times \vec{B}) - \nu_{en}\vec{v}_e,$$

$$0 = \frac{e}{m_i}(\vec{E} + \frac{1}{c}\vec{v}_i \times \vec{B}) - \nu_{in}\vec{v}_i.$$

[10 min.] **(2a)** Show that the current density $\vec{j}$ is given by

$$\vec{j} = \sigma_\perp \vec{E} - \sigma_H \frac{\vec{E}\times\vec{B}}{B}. \tag{2.1}$$

Derive $\sigma_\perp$ and σ_H.

[20 min.] **(2b)** Near $h \approx 110$ km altitude, there is a layer (the E layer) where the conductivity σ is much larger than that on either side of the layer in the equatorial ionosphere. (See Fig. 3.) Assuming $\sigma = 0$ on both sides of the layer, find E_y in terms of E_0. Also find the total current induced in the x direction in terms of E_0.

* * *

PROBLEM 3: *Waves and instabilities* [10 minutes total].

Consider a plasma for $-\infty < x < \infty$. Assume that a perturbation

$$\delta\varphi(x,t) = \delta\widehat{\varphi}(x)\exp(-i\omega t)$$

has the following *two* asymptotic solutions as $|x| \to \infty$:

$$\lim_{|x|\to\infty} \delta\widehat{\varphi}(x) \to \exp(\pm ix^2/\omega).$$

Indicate which one of the solutions satisfies the proper boundary condition and explain why.

* * *

PROBLEM 4: *Experimental* [20 minutes total].

Controlling plasma impurities (the non-fuel component) is an important problem in controlled thermonuclear fusion research. For a tokamak plasma:

[3 min.] **(4a)** List reasons why controlling impurities are important.

[5 min.] **(4b)** List possible sources of impurities. How are they introduced into the plasma?

[6 min.] **(4c)** What are the experimental techniques being used to measure the impurity levels?

[6 min.] **(4d)** List experimental procedures and approaches that are being pursued to *control* the impurity levels. Can you think of situations in which impurities are actually useful?

* * *

PROBLEM 5: *Alfvén waves* [30 minutes total].

A packet of shear-Alfvén waves ($\omega^2 = k_\parallel^2 c_A^2$) propagates in a cold plasma along the axis of an axisymmetric field. Proceeding along the axis from Region I to Region II, the plasma density and the zero-order magnetic field change—*very slowly*—from (ρ_I, B_I) to (ρ_{II}, B_{II}). (Near the axis, let ρ and B be considered to be functions of distance along the axis only.) What are the ratios of the rms amplitudes for the electric and magnetic fluctuating fields in Region II compared to their respective values in Region I?

* * *

PROBLEM 6: *Mirrors* [15 minutes total].

Consider a plasma confined in a magnetic mirror geometry. Suppose the electron and ion distribution functions are known (but different) functions of energy and magnetic moment. Outline the analytic procedure necessary to calculate the electrostatic potential variation along a magnetic field line. (If your answer involves integrals, be sure to specify the limits of integration explicitly.)

* * *

PROBLEM 7: *Kinetic theory* [50 minutes total].

This problem relates to neutral beam heating in typical tokamak experiments. Consider the slowing down and scattering of an 80 keV beam of deuterium ions in a neutral electron-proton plasma, where the electron and hydrogen temperatures are given by $T_e = T_i = 1$ keV, and the plasma density is $n = 10^{14}\mathrm{cm}^{-3}$. The evolution of the deuterium distribution function f_d is described rather well for $t > 0$ by the following equation:

$$\frac{\partial f_d}{\partial t} = \frac{1}{\tau_s}\left[\underset{\text{(a)}}{\frac{1}{v^2}}\,\underset{\text{(b)}}{\frac{\partial}{\partial v}}[(v^3+v_c^3)f_d] + \underset{\text{(c)}}{\frac{1}{2}\frac{m_i}{m_d}\frac{v_c^3}{v^3}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}f_d}\right] + S(v,\mu), \qquad (7.1)$$

where $S(v,\mu)$ is a time-independent source function (turned on at $t = 0$) for the deuterium, and

$$v_c^3 \stackrel{\text{def}}{=} \frac{3\sqrt{\pi}}{4}\frac{m_e}{m_i}v_{te}^3, \qquad \tau_s \stackrel{\text{def}}{=} \frac{4\pi}{nL}\frac{3\sqrt{\pi}}{4}\frac{m_e}{m_d}v_{te}^3, \qquad L \stackrel{\text{def}}{=} \frac{e^4\ln\Lambda}{4\pi\epsilon_0^2 m_d^2}.$$

[5 min.] **(7a)** Interpret each of the terms (a)–(c) in Eq. (7.1). What approximations were used in obtaining this equation from the Landau collision integral? In what region of velocity space does the equation fail to represent adequately the evolution of the deuterium distribution?

[10 min.] **(7b)** This equation may be solved in the finite region bounded by 80 keV, or $v < v_0$, where $m_d v_0^2/2 \stackrel{\text{def}}{=} 80\,\mathrm{keV}$. Show that, for appropriate boundary and initial conditions, the solution to this equation, if it exists, is unique. Explain in physical terms the most general conditions that must be imposed to ensure uniqueness.

[5 min.] **(7c)** Suppose that initially $f_d = 0$ and that the deuterium is injected at the rate $\dot{N}$ $\mathrm{cm}^{-3}\mathrm{sec}^{-1}$ at exactly 80 keV in exactly the z direction. Use contour plots to sketch the deuterium distribution both for times short compared to τ_s and for times long compared to τ_s. Make and indicate any reasonable, physical assumptions that you need to produce these sketches. *(Continued on next page.)*

[10 min.] **(7d)** What is the total steady-state heating power? How is this power apportioned between the heating power P_e to electrons and the heating power P_i to ions? You may leave your answer in terms of elementary integrals.

[20 min.] **(7e)** Suppose, for simplicity, that we are interested only in the pitch-angle-averaged distribution, namely $F(v,t) = \int d\mu\, f(v,\mu,t)$. Use the method of characteristics to solve for $F(v,t)$. Sketch $F(v,t)$. [***Hint:*** The algebra is somewhat easier if you solve for $g \stackrel{\text{def}}{=} (v^3 + v_c^3)F(v,t)$.] Describe briefly (one sentence suffices) how you would solve for $f(v,\mu,t)$.

* * *

PROBLEM 8: *MHD* [10 minutes total].

An intuitive form of the plasma (or "fluid") part δW_f of δW is as follows:

$$\delta W_f = \tfrac{1}{2}\int_{\text{plasma}} d\vec{x} \Bigg[\underbrace{\frac{|\vec{Q}_\perp|^2}{\mu_0}}_{(1)} + \underbrace{\frac{B^2}{\mu_0}|\vec{\nabla}\cdot\vec{\xi}_\perp + 2\vec{\xi}_\perp\cdot\vec{\kappa}|^2}_{(2)} + \underbrace{\gamma P |\vec{\nabla}\cdot\vec{\xi}|^2}_{(3)} \underbrace{- 2(\vec{\xi}_\perp\cdot\vec{\nabla}P)(\vec{\kappa}\cdot\vec{\xi}_\perp^*)}_{(4)} \underbrace{- j_\parallel(\vec{\xi}_\perp^*\times\hat{b})\cdot\vec{Q}_\perp}_{(5)} \Bigg].$$

[5 min.] **(8a)** Briefly state the physical significance of each of the five terms, including which of the MHD waves or instabilities it contributes to in tokamaks.

[5 min.] **(8b)** Which two terms balance to determine the ballooning-mode stability of tokamaks? Estimate the stability criterion.

* * *

PROBLEM 9: *Applied math* [25 minutes total].

Consider the recurrence relation in n

$$n^2 Q_n(x) = (2n-1)\frac{\sqrt{1+x^2}}{x} Q_{n-1}(x) - Q_{n-2}(x),$$

where x is a real, positive parameter and n is an integer.

[5 min.] **(9a)** How many independent solutions does this recurrence relation have? Justify your answer.

[20 min.] **(9b)** Suppose you are given the initial conditions

$$Q_{-1}(x) = 1,$$
$$Q_0(x) = \frac{\sinh^{-1} x}{x}.$$

With such initial conditions, it can be shown that for $n \to +\infty$

$$Q_n(x) \to \left(\frac{\pi}{2x}\right)^{1/2} \frac{1}{\Gamma(n+3/2)} \left(\frac{x}{1+\sqrt{1+x^2}}\right)^{n+1/2}.$$

(Don't try to prove this.) Describe how to evaluate $Q_n(x)$ ***numerically*** for large positive n ($n \sim 100$, say) and fixed x in such a way as to minimize the error in the result.

Hint: There is a "wrong" way of implementing such a method in which the error grows exponentially with n.

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DEPARTMENT OF ASTROPHYSICAL SCIENCES

PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION

OCTOBER 1988

PART I - OCTOBER 17

- Answer all problems

- NOTE: Choose either Set (A and B) or Set (C and D) for Problem #4. The time of each set is the same.

- The total time allotted for this day is 3 3/4 hours. Scores on questions will be weighted in proportion to the allotted time.

- Start each numbered problem on a new page. Put your name on every page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem __ " and sign your name.

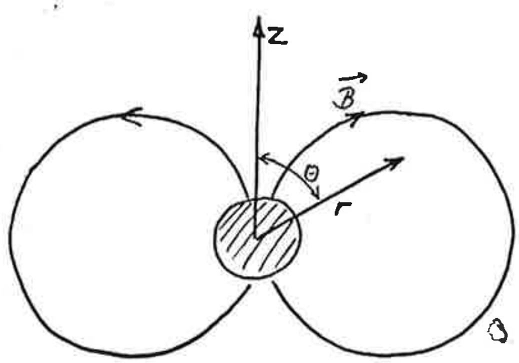
GOOD LUCK!

PROBLEM #1 Particle Orbits 40 mins.

Consider particle drifts in the Earth's magnetic field, which we model as a dipole located at the Earth center with the north pole pointing in the z-direction. The magnetic potential of the dipole is given by

$$V = - M \cos \theta / r^2,$$

where M is the magnetic moment of the Earth and (r,θ) is the radius and polar angle (using the spherical coordinates r,θ,ψ).



1. 5 mins. Derive the components of the magnetic field $\vec{B}(r,\theta,\psi)$ from $\vec{B} = \vec{\nabla}V$ in the spherical coordinates.

2. 10 mins. Consider the motion of a single particle with charge q and mass m, located at $(r,\theta,\psi) = (R_o,\pi/2,\psi_o)$ and with initial velocity given by $(v_\perp,v_\parallel) = (v_o\ ,0)$, where $\perp$ and $\parallel$ correspond to perpendicular and parallel to $\underset{\sim}{B}$ field. Find the drift velocity in the ψ direction and calculate the current associated with the drift motion. Find the induced magnetic field at the Earth's center.

3. 10 mins. The gyromotion of the particle represents a dipole moment. Calculate the magnetic field at the Earth's center due to particle gyromotion around the magnetic field. Find the total magnetic field induced at the origin.

4. 15 mins Add uniform magnetic and electric fields $\underline{B} = B_o \hat{z}$, $\underline{E} = E_o \hat{x}$ to the dipole magnetic field. Show the orbit of a particle whose coordinates are initially $(x,y,z) = (x_o, y_o, 0)$ with $(v_\perp, v_\parallel) = (v_o, 0)$ is given by

$$x - x_o = \left(\frac{\mu M}{qE_o}\right) [(x^2 + y^2)^{-3/2} - (x_o^2 + y_o^2)^{-3/2}]$$

where μ is the magnetic moment of the particle.

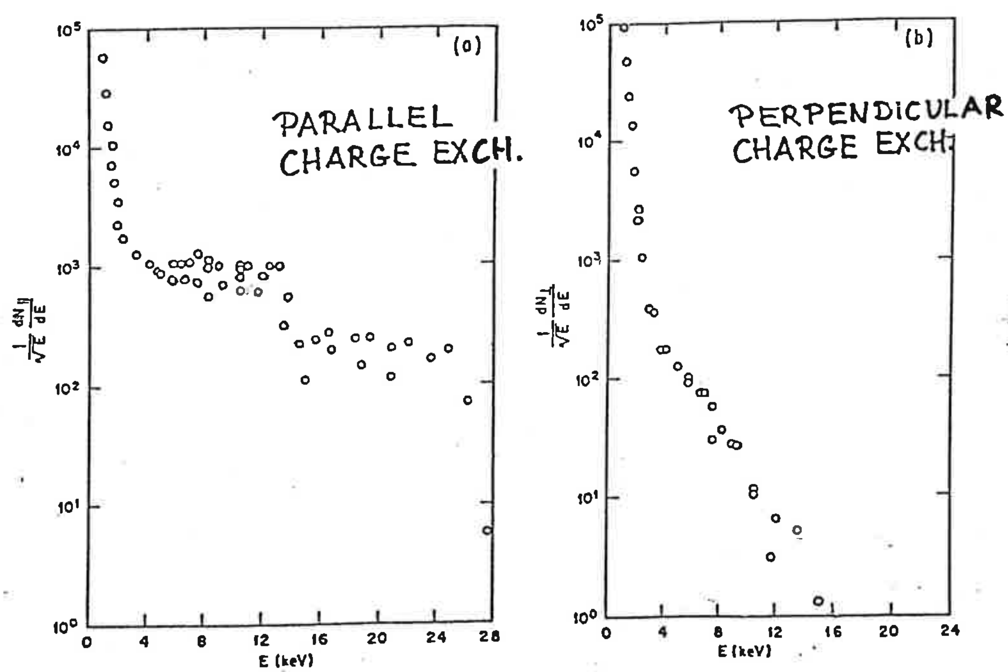
PROBLEM # 2 EXPERIMENTAL METHODS - NEUTRAL BEAM INJECTION

35 mins

1. 5 mins Make a sketch of a neutral injection system for a tokamak. List and describe the main functions of each of the components.

2. 15 mins The system has to deliver a certain beam current I_b (e.g., 20 A) of 60 keV deuterons into an area A (e.g., 10 × 10 cm). Explain in a semi-quantitative fashion, how you would dimension each of the components. How do you focus the beam?

3. 5 mins Discuss the features and wiggles of the parallel and perpendicular charge exchange spectra from a tokamak plasma into which a 26 keV neutral beam was injected.



4. 10 mins. Discuss the atomic processes which determine the attenuation of the neutral beam in the plasma. Sketch the dependence of the atomic processes on the energy of the injected neutral particles.

PROBLEM #3 SINGLE PARTICLE MOTIONS (30 mins)

20 mins.

a). A charged particle in a uniform magnetic field, $\vec{B} = \hat{z}B_o$, is subject to a weak electric field, $\vec{E} = \hat{x}\, f(t) + \hat{y}g(t)$. When Fourier analyzed over a cyclotron period, $\vec{E}(\Omega) = \sum_n (\hat{x}\tilde{f}_n + \hat{y}\tilde{g}_n)\, e^{-in\Omega t}$, $\tilde{f}_n(\Omega)$ and $\tilde{g}_n(\Omega)$ have no components at $n = 0$ or $n = 1$, corresponding to zero-frequency and the cyclotron fundamental, respectively. Show that the orbit is periodic.

10 mins

b). Would this same conclusion be valid in the case $\vec{E} = -\nabla\psi(x,y,t)$, but still with $\tilde{\psi}_n(x,y,\Omega) = 0$, $n = 0$ or 1, referring again to Fourier analysis over a cyclotron period? Defend your answer in a qualitative manner. Is resonant acceleration possible in this case? Explain.

PROBLEM #4 CHOOSE TO DO EITHER SET {A AND B} OR SET {C AND D}.

4A. MHD PROBLEM 20 mins.

Paramagnetism and Diamagnetism in Tokamak Equilibrium

Consider a "straight tokamak" i.e., neglect toroidal effects. Show that depending on the value of β_θ, defined by

$$\beta_\theta = \frac{16\pi^2}{\mu_o I_p^2} \int_o^a prdr$$

the tokamak will be either paramagnetic or diamagnetic with respect to the "toroidal" field. Here I_p is the "toroidal" current, p(r) is the plasma pressure, and a is the minor radius.

PROBLEM #4B. WAVES PHYSICS 30 mins.

The wave equation

$$\vec{k} \times (\vec{k} \times \vec{E}) \frac{c^2}{\omega^2} + \vec{K} \cdot \vec{E} = 0$$

where $\vec{K}$ is the dielectric tensor, can be approximated for an electron plasma wave in a dilute plasma with a very strong magnetic field as

$$\begin{pmatrix} 1 - n_\parallel^2 & 0 & n_\perp n_\parallel \\ 0 & 1-n^2 & 0 \\ n_\perp n_\parallel & 0 & -(\omega_{pe}^2/\omega^2) - n_\perp^2 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

Here we assumed $n_x = ck_x/\omega = n_\perp$, $n_y = 0$, and $n_z = n_\parallel$. In the limit of $n_\perp^2$, $n_\parallel^2 >> 1$, find the

(a) Wave dispersion relation

(b) Wave phase velocities, $v_{ph}(\perp)$ and $v_{ph}(\parallel)$

(c) Wave group velocities, $v_g(\perp)$ and $v_g(\parallel)$

(d) Wave energy density and power flux, $P(\perp)$ and $P(\parallel)$

(e) Wave impedance, B_y/E_z.

2ND CHOICE SET

PROBLEM #4C. NUMERICAL ANALYSIS 10 mins.

A function $y(x)$ satisfies

$$\frac{dy}{dx} = f(y) \quad , \quad y(0) = y_o$$

Outline a scheme for calculating $y(x)$ numerically. Estimate the error, i.e. the difference between the computed and true values of $y(x)$ for the case $f(y) = y$.

PROBLEM 4D. INSTABILITIES: MAGNETIC ISLANDS 40 mins.

Consider a tokamak in cyclindrical approximation with

$$\vec{B} = B_0[\hat{\phi} + \frac{r}{Rq(r)}\hat{\theta}]$$

Introduce a perturbation $\delta\vec{B} = \hat{r}b\cos(m\theta - n\phi)$ ($\delta\vec{B}$ must also have other components, but they can be ignored).

a. Find the equations for the field lines, i.e. calculate the quantities, $d\theta/d\phi$ and $dr/d\phi$, where θ and r are coordinates of the field line.

b. Integrate to find the flux surfaces $r(\theta,\phi)$ near a surface r_s with $q(r_s) = m/n$.

c. Find the magnetic island width Δr.

d. Describe what happens if there are two such perturbations (m_1,n_1) and (m_2,n_2) and find an approximate critical threshold level for the perturbation amplitudes (Chirikov overlap condition).

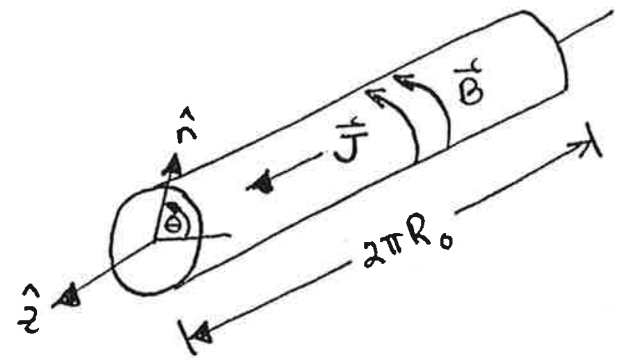
PROBLEM #5 KINETIC THEORY 25 mins

15 mins: Show that if f obeys the Klimontovich Equation, then entropy is conserved. State clearly any assumptions or facts that you use.

10 mins: Very briefly, discuss the significance of this conservation. In particular, contrast the Klimontovich equation to an equation in which entropy is not conserved.

PROBLEM #6 IDEAL MHD PROPERTIES OF THE Z-PINCH 45 mins.

The linear $\hat{z}$ pinch is like a "straight" (non-toroidal) tokamak with no equilibrium toroidal field (B_z), and with circular cross section.



Cylindrical coordinates (r,θ,z). $\vec{B} = (0,B_\theta,0)$.

PART A: Equilibrium

(i) Write down the equilibrium equation satisfied by $B_\theta(r)$ and the scalar pressure $p(r)$.

(ii) If $B_\theta(r)$ is given by the profile

$$B_\theta(r) = \frac{\mu_o I_o}{2\pi} \frac{r}{r^2+r_o^2}$$

then what is the equilibrium pressure $p(r)$?

PART B: Stability to m=0 perturbations

Consider a small displacement $\vec{\xi}(r)\, e^{ikz}$ (no θ variation), where $\vec{\xi}(r) = [\xi_r(r), 0, \xi_z(r)]$, (no component in the $\hat{\theta}$ direction). The change in potential energy for infinitesimal displacement $\vec{\xi}$ is given by

$$\delta W = \int_o^a r dr \left\{ B_\theta^2 k^2 |\xi_z|^2 + 2 \frac{p'}{r} |\xi_r|^2 + B_\theta^2 r^2 \left|\frac{\xi'_r}{r}\right|^2 \right.$$

$$+ rikB_\theta{}^2 [\xi_z (\frac{\xi_r^*}{r})' - \xi_z^* (\frac{\xi_r}{r})']$$

$$\left. + \gamma p [\frac{1}{r} (r\xi_r)' + ik\xi_z][\frac{1}{r} (r\xi_r^*)' - ik\xi_z^*] \right\}$$

where prime ()' denotes differentiation with respect to r, $\partial/\partial r()$, and star ()* denotes complex conjugate.

(iii) What is the minimizing value of $\xi_z(r)$, expressed in terms of the other variables?

(iv) Using the result found in (iii), can you find a necessary and sufficient condition for the $B_\theta(r)$ and p(r) profiles to be stable?

DEPARTMENT OF ASTROPHYSICAL SCIENCES

PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION

OCTOBER 1988

PART II - OCTOBER 18

- Answer all problems

- Choose either Set {A and B} or Set {C, D and E} for Problem #4. The time for each set is the same.

- The total time allotted for this day is 3 3/4 hours. Scores on questions will be weighted in proportion to the allotted time.

- Start each numbered problem on a new page. Put your name on every page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem __ " and sign your name.

GOOD LUCK!

PROBLEM #1 WAVES AND INSTABILITIES QUICKIE 10 mins.

Consider the injection of a neutral beam into a plasma perpendicular to a uniform magnetic field. Assuming the neutral particles are ionized along the beam path, show that polarization charges and hence an electric field appear in the beam. Discuss the stability of the beam under the influence of the electric field.

PROBLEM #2 KINETIC THEORY 50 mins.

This problem relates to neutral beam heating in typical tokamak experiments. Consider the slowing down and scattering of an 80 keV beam of deuterium ions in a neutral electron-proton plasma, where the electron and hydrogen temperatures are given by $T_e = T_i = 1$ keV, and the plasma density is $n = 10^{14} cm^{-3}$. The evolution of the deuterium distribution function, f_d, is described rather well for $t > 0$ by the following equation:

$$\frac{\partial f_d}{\partial t} = \frac{1}{\tau_s}\left\{\frac{1}{v^2}\frac{\partial}{\partial v}\left[(v^3 + v_c^3)f_d\right] + \frac{1}{2}\frac{m_i}{m_d}\frac{v_c^3}{v^3}\frac{\partial}{\partial \mu}(1 - \mu^2)\frac{\partial}{\partial \mu}f_d\right\} + S(v,\mu) \qquad (*)$$

where $S(v,\mu)$ is a time-independent source function (turned on at $t = 0$) for the deuterium, and

$$v_c^3 = \frac{3\sqrt{\pi}}{4}\frac{m_e}{m_i}v_{Te}^3, \quad \tau_s \equiv \frac{4\pi}{nL}\frac{3\sqrt{\pi}}{4}\frac{m_e}{m_d}v_{te}^3, \quad L \equiv \frac{e^4 \ln\Lambda}{4\pi\epsilon_o^2 m_d^2}$$

a) 5 mins. Interpret each of the terms in Eq. (*). What approximations were used in obtaining this equation from the Landau collision integral? In what region of velocity space does the equation fail to represent adequately the evolution of the deuterium distribution?

b) 15 mins. This equation may be solved in the finite region bounded by 80 keV, or $v < v_o$, where $m_d v_o^2/2 = 80$ keV. Show that, for appropriate boundary and initial conditions, the solution to this equation, if it exists, is unique. Explain in physical terms the most general conditions that must be imposed to ensure uniqueness. HINT: choose an appropriate weighting factor in constructing your quadratic form.

c) 10 mins. Suppose that initially $f_d = 0$ and that the deuterium is injected at the rate $N cm^{-3} sec^{-1}$ at exactly 80 keV in exactly the z-direction. What is the total steady state heating power? How is this power apportioned between the heating power to electrons, P_e, and the heating power to ions, P_i. You may leave your answer in terms of elementary integrals. In general, injection is such that $v_o > v_c$. However, for the limit $v_o \ll v_c$, what is the ratio of power absorbed by electrons to power absorbed by ions.

d) 20 mins. Suppose, for simplicity, that we are interested only in the pitch-angle averaged distribution, namely $F(v,t) = \int d\mu \; f(v,t)$. Use the method of characteristics to solve for $F(v,t)$. Sketch $F(v,t)$. (HINT: the algebra is somewhat easier if you solve for $g \equiv (v^3 + v^3_c) \; F(v,t)$). Describe briefly (one sentence suffices) how you would solve for $f(v,\mu,t)$.

PROBLEM #3 WAVES CYCLOTRON HARMONIC DAMPING 40 mins.

The dispersion relation for electrostatic oscillations in a uniform collisionfree magnetized plasma is

$$k_x^2 + k_z^2 + 4\pi \sum_n \sum_s \frac{q_s^2}{m_s} \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} 2\pi v_\perp dv_\perp \frac{J_n^2(z)}{\omega - k_z v_z - n\omega_c} \left(k_x \frac{n}{z} \frac{\partial f}{\partial v_\perp} + k_z \frac{\partial f}{\partial v_z}\right) = 0$$

where $k_y = 0$, $f = f_o(v_\perp, v_\parallel)$, $z_s = k_x v_\perp/\omega_{cs}$, $\omega_{cs} = q_s B/m_s c$.

a) 15 mins. Taking the cold plasma limit, put this dispersion relation into the form

$$D = k_x^2\left(1 + \sum_s \frac{\omega_{ps}^2}{\omega_{cs}^2 - \omega^2}\right) + k_z^2\left(1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\right) = 0$$

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \left[1 - \frac{(z/2)^2}{1!(n+1)} + \frac{(z/2)^4}{2!(n+1)(n+2)} - \ldots\right]$$

b) 25 mins. The cold-plasma relation $D = \ldots = 0$ in Part (a) describes, among others, lower hybrid waves. Assume that the parameters are such that this dispersion relation is satisfied by $\omega \simeq 3\omega_{ci}$ and assume now that the ion temperature is low but finite. Rewrite the dispersion relation to include 3rd harmonic ion cyclotron damping. Assume $|k_x| \gg |k_z|$. Finally, write down an equation for the damping rate.

PROBLEM #4 CHOOSE TO DO EITHER SET (A & B) OR SET (C, D & E).

35 mins.

4A. INTRODUCTORY PLASMA PHYSICS 10 mins.

a) 5 mins. Radiation from a supernova reaches the earth's ionosphere. Will these waves be appreciably Landau-damped?

b) 5 mins. One way to define resistivity η is by the equation

$$\underline{E} = \eta \underline{j}$$

However, this is inappropriate for perpendicular Spitzer resistivity. Why? What is the appropriate way (i.e. by what physical mechanism) to define $\eta_{\perp}$.

4B. EXPERIMENTAL METHODS 25 mins.

An omegatron is a mass spectrometer for ions which uses crossed electric and magnetic fields defined by

$$\underline{E} = \underline{x} E_o \sin \omega t$$

and

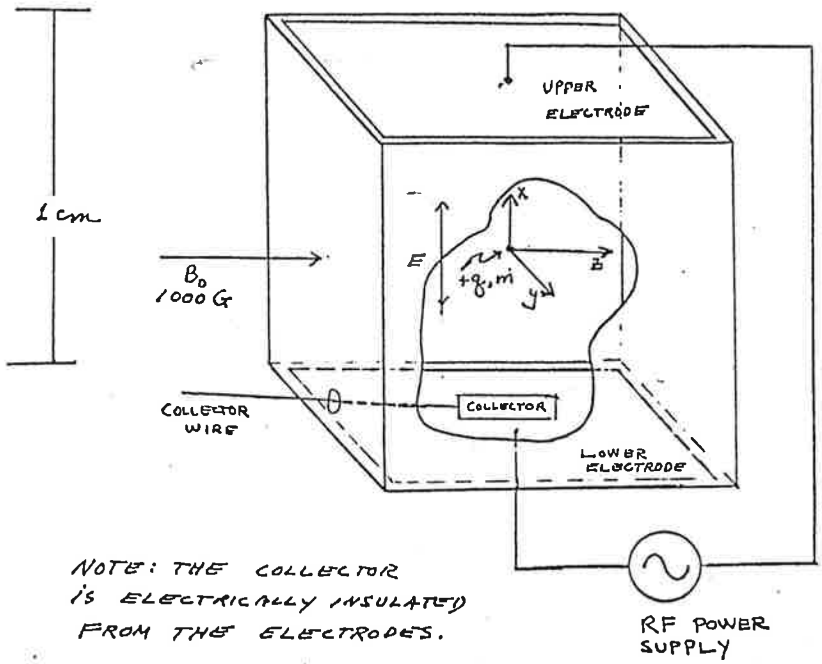
$$\underline{B} = \underline{z}\, B_o .$$

a) Solve the equations of motion of a "resonant" (i.e. $\omega = \Omega_c$) particle. Sketch the orbit. Assume that $v_\parallel = 0$.

b) Consider the operation of such a device with field strengths $E_o = 10$ V/cm and $B_o = 10^3$G and dimensions as shown in the figure.

i) Calculate the resonant frequencies.

ii) What are the total path lengths traversed by an H^+ ion and an Ar^+ ion as they are resonantly (separately) pumped from the axis to the collector?



NOTE: THE COLLECTOR IS ELECTRICALLY INSULATED FROM THE ELECTRODES.

SECOND CHOICE SET

PROBLEM #4C HEAT FLUX PROBLEM 20 mins.

It is often important to estimate a heat flux (per unit area) of a plate exposed to a plasma of density n, T_e, and ion mass, M. Show that the main term of the heat flux is proportional to $nT_e^{3/2}M^{-1/2}\ln(M/m)$, where m is the electron mass. For simplicity, you can assume that $T_i \ll T_e$. The plate is assumed to be at the floating potential. (Hint: A simple application of Langmuir probe theory).

PROBLEM #4D EXPERIMENTAL QUICKIE 10 mins.

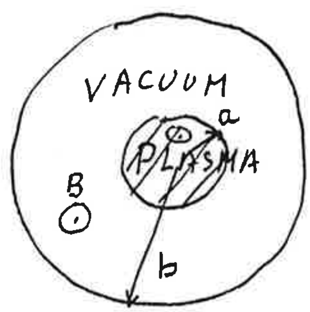
A 400 eV electron is moving in a simple torus in the direction of the toroidal field which is 5 kG at the orbit radius of 60 cm. State the direction and magnitude of an additional magnetic field needed to keep the electron in the circular orbit.

PROBLEM #4E EXPERIMENTAL QUICKIE 5 mins.

List three distinct methods used to measure electron temperature in plasmas with thermonuclear type parameters. Do the same for low temperature ($T \sim 5$eV) plasmas.

PROBLEM #5 MHD PROBLEM 45 mins.

An infinite cylinder of radius b contains a uniform axial magnetic field $\underline{B} = B_o \hat{z}$ and a concentric uniform cold infinitely conducting plasma of initial radius a_o and density n_o. The shell of the cylinder is also infinitely conducting and the region between the plasma and this shell is a vacuum.



The plasma is slowly heated to a temperature T so that it stays in MHD equilibrium.

1. 5 mins. What will happen? What is conserved?

2. 15 mins. Write an algebraic equation which determines the final radius of the plasma in terms of its initial parameters and T (do not solve it).

3. 10 mins. Is there an upper limit to the temperature for which the plasma is confined?

a) If so what is it?

b) If not, determine the plasma radius and shell thickness for large T.

4. 15 mins. Repeat the first and second part of the problem if the shell is not infinitely conducting but is constrained by external circuitry to carry a constant surface current (The plasma must not touch the wall).

PROBLEM #6 APPLIED MATH 45 mins.

This question explores the limits of an asympotic representation.

a) 5 mins. Show that

$$f(x) = \int_0^\infty \frac{e^{-xt}}{1+t^2}\, dt \qquad (\text{for } x > 0) \qquad \text{(i)}$$

satisfies the equation

$$\frac{d^2 f}{dx^2} + f = \frac{1}{x} \qquad (\text{for } x > 0) \qquad \text{(ii)}$$

with the boundary conditions $f \to 0 \quad x \to \infty$

$f = \pi/2 \quad x = 0$

b) 10 mins. Find a asymptotic series for $f(x)$ as $x \to \infty$

HINT: $\int_0^\infty t^{2N}\, e^{-xt} dt = \frac{1}{x}\, \frac{(2N)!}{x^{2N}}$

Is this series convergent?

c) 15 mins. Show that the remainder after N terms of the series is

$$\epsilon_N = (-1)^{N+1} \int_0^\infty \frac{t^{2(N+1)}}{1+t^2}\, e^{-xt}\, dt \qquad \text{(iii)}$$

Also show that $|\epsilon_N| < \frac{1}{x}\, \frac{[2(N+1)]!}{x^{2(N+1)}}$