

DEPARTMENT OF ASTROPHYSICAL SCIENCES

PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION

MAY 14, 1990

PART I

- Answer all problems.
- The total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to the allotted time.
- Start each numbered problem on a new page. Put your name on every page.
- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem ____ " and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- Books, calculators, and other aids are not permitted. Barbara Sarfaty will provide an NRL Formulary.

Introductory Plasma Physics (particle orbits, etc.)

20 minutes

This problem deals with basic drift motions:

#1A - 10 minutes

a) From simple equilibrium conditions (scalar pressure, no temperature gradient, and $\mathbf{B} = B_0 \hat{\mathbf{z}}$) derive an expression for the diamagnetic drift velocity. Explain why this is called a "diamagnetic" velocity.

#2A - 10 minutes

b) From a simple single particle orbit picture, first obtain the $\mathbf{E} \times \mathbf{B}$ drift velocity and then go to next order in ω/Ω (with Ω being the cyclotron frequency and ω being a measure of the time-rate-of-change of the E-field), to obtain the polarization drift velocity.

Short questions

25 minutes

#3A - 15 minutes

The word "resonance" is unfortunately used with several meanings in plasma physics. Distinguish what is meant by a

- a) wave-particle resonance,
- b) mode-conversion resonance, and
- c) cavity resonance.

#4A - 10 minutes

What is the difference between the drift-kinetic and gyro-kinetic equations?

Single-Particle Orbits

60 minutes

#5A

5 minutes

a) What condition must ψ satisfy such that the equation $\psi(\vec{r}) = \text{constant}$ describes a magnetic surface?

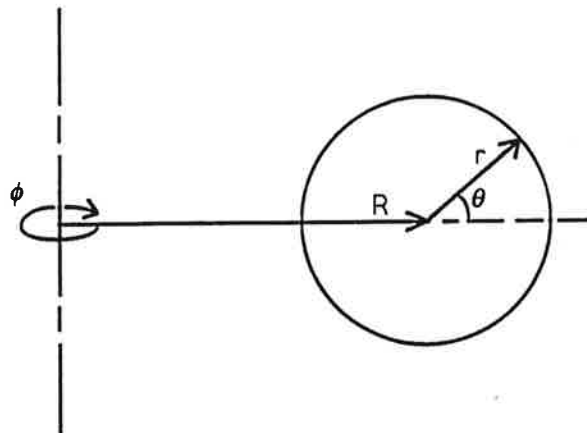
5 minutes

b) Take model fields for tokamak geometry, in toroidal coordinates (r, θ, ϕ) ,

$$B_\phi = \frac{B_t(r) R}{R + r \cos\theta}$$

$$B_\theta = \frac{B_p(r) R}{R + r \cos\theta}$$

$$B_r = 0$$



Find a suitable $\psi(\vec{r})$ for this case.

30 minutes

c) At $t = 0$, a particle at some r, θ, ϕ has finite values for v_\perp and v_\parallel . Neglecting finite Larmor radius effects, find the equation that will describe the poloidal projection of the particle trajectory. Indicate how and why this curve may differ from a magnetic surface.

10 minutes

d) Say that there is an axisymmetric electrostatic field described by the potential $V(r, \theta)$ also present. Modify the equation in Part (c) to accommodate this change.

10 minutes

e) Indicate in a sketch how the turning points of trapped particles would be affected by a uniform vertical field, $\vec{E} = \hat{z} E_0$.

Applied Math
30 minutes

#6A

1. Fusion reaction cross sections can be approximated at low energies by:

$$\sigma(E) = \frac{s(E)}{E} \exp\left(-2\left(\frac{E_0}{E}\right)^{1/2}\right)$$

$s(E)$ is a function deduced from experiments - typically

$$\frac{1}{s} \frac{ds}{dE} \sim O\left(\frac{1}{E_0}\right)$$

In a star or a tokamak the reacting species have a Maxwellian distribution,

$$f dE = \frac{2}{\sqrt{\pi}} \left(\frac{E}{kT}\right)^{1/2} \exp\left(-\frac{E}{kT}\right) \frac{dE}{kT}.$$

The reaction rate (Fusion reactions per second) is,

$$\lambda(T) = \langle \sigma v \rangle = \int_0^\infty \sigma(E) \sqrt{\frac{2E}{m}} f(E) dE$$

therefore

$$\lambda(T) = \left(\frac{8}{m\pi}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty s(E) \exp\left[-\frac{E}{kT} - 2\left(\frac{E_0}{E}\right)^{1/2}\right] dE.$$

Evaluate $\lambda(T)$ for $kT \ll E_0$ to lowest order.

#7A

Introductory Plasma Physics (45 minutes)

Consider a semi-infinite, unmagnetized, collisionless plasma, bounded by two parallel metal walls. The electrons have a temperature T_e , mass m_e , and density n_e ; the ions (mass m_i , density n_i) are **isothermal** with a temperature T_i . Electrons, being more mobile than ions, will rush to the walls, leaving behind a plasma slightly rich in ions. The electron flow will now diminish because of the resultant electric field which retards their losses. A potential drop of $\sim 4T_e$ will exist in a region roughly $5 \lambda_D$ in thickness near the walls. A steady state will soon occur with small residual electron and equal ion losses to the walls, provided that a **volumetric source term**, $S(x)$, balances the lost plasma particles.

Goals of this problem are:

- to derive the velocity of ions, $v(x)$, as they approach the metal walls; and
- to refine the estimate of electric field penetration **into** the plasma, where $n_e = n_i \equiv n$.

This can be done using the 2-fluid plasma equations.

- Write down the ion continuity equation under the assumption of steady state. (What term balances the particle fluxes?)
- Write down the ion momentum equation under the assumption of steady state. (What term takes into account the "birth" of particles with no momentum, hence a drag on the velocity $v(x)$?)
- Write down the expression for the spatial dependence of the electron density under the assumption that it is solely determined by the Boltzmann relation.
- With the definitions $C_s^2 \equiv (kT_e + kT_i)/m_i$ and $M \equiv v(x)/C_s$, use the above equations to show that

$$dM/dx = (S/nC_s) [(1+M^2)/(1-M^2)].$$

Note that when $M \rightarrow 1$, $dM/dx \rightarrow \infty$. This corresponds to the condition at the sheath. The ions are rapidly accelerated away from the electrons; quasineutrality fails.

- Using the definition that $n(x) = n(0)$ where $M=0$ (at the symmetry plane between the two walls),

show how the density and potential depend on M .

In particular, find that at the sheath, $n(s) = n(0)/2$ and $\phi(s) = \frac{kT_e}{e} \ln 2$.

MHD Long Problem

60minutes

#8A

Application of energy principle to $m \gg 1$ ideal MHD ballooning modes in an axisymmetric mirror plasma

Let us consider ideal MHD ballooning modes with short perpendicular wavelengths in a mirror plasma. That is,

$$\delta \tilde{\xi} = \hat{\delta \tilde{\xi}}(\ell) \exp(im\theta),$$

where $\delta \tilde{\xi}$ is the MHD fluid displacement vector and $|m| \gg 1$. Here we use the (ψ, θ, ℓ) coordinates. Thus,

$$\tilde{B} = \nabla \psi \times \nabla \theta,$$

ψ is the flux coordinate, θ is the azimuthal symmetry coordinate, and ℓ is the coordinate along \tilde{B} . Figure 1 illustrates the mirror geometry.

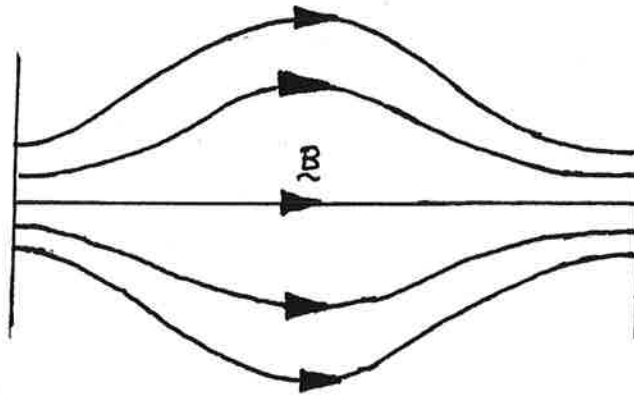


Fig. 1

$\delta \tilde{\xi}$ is assumed to have the fixed boundary conditions, i.e.

$$\hat{\delta \tilde{\xi}}(\pm \ell_0) = 0,$$

where $\pm l_0(\psi)$ are the positions of the two conducting end plates.

The energy principle is given as follows

$$\delta W(\psi) = \int_{-l_0}^{l_0} \frac{d\ell}{B} \left[|\nabla \times (\delta \xi_{\perp} \times B)|_{\perp}^2 + B^2 |\nabla \cdot \delta \xi_{\perp} + 2\delta \xi_{\perp} \cdot \kappa|^2 \right.$$

$$\left. -8\pi (\delta \xi_{\perp} \cdot \nabla P) (\delta \xi_{\perp}^* \cdot \kappa) + 4\pi \gamma P |\nabla \cdot \delta \xi|^2 \right].$$

Here, $P = P(\psi)$ is (for simplicity) the isotropic equilibrium pressure, γ is the ratio of the specific heat, $\beta = 8\pi \gamma P/B^2 \sim 0(1)$ and $\kappa = (B \cdot \nabla B)/B^2$ is the curvature.

(I) Minimizing δW w.r.t. $\delta \xi_{\parallel}$

$$[\text{Hint: } \nabla \cdot \delta \xi = \nabla \cdot \delta \xi_{\perp} + B \frac{\partial}{\partial \ell} (\delta \xi_{\parallel}/B)]$$

15 minutes

(I.1) Show that

$$\frac{\partial}{\partial \ell} (\nabla \cdot \delta \xi) = 0.$$

15 minutes

(I.2) Show that

$$\nabla \cdot \delta \xi = \left[\int_{-l_0}^{l_0} \frac{d\ell}{B} (\nabla \cdot \delta \xi_{\perp}) \right] \left[\int_{-l_0}^{l_0} \frac{d\ell}{B} \right]^{-1} \equiv \overline{\nabla \cdot \delta \xi_{\perp}}.$$

(II) Minimizing δW w.r.t. $\delta \xi_{\perp}$

Expanding

$$\hat{\xi}_{\perp} = \hat{\xi}_{\perp 0} + \hat{\xi}_{\perp 1} + \dots,$$

where

$$|\hat{\xi}_{\perp n}| / |\hat{\xi}_{\perp 0}| \sim O\left(\frac{1}{m}\right)^n.$$

15 minutes

(II.1) Denoting $\mathbf{k}_{\theta} = m \nabla \theta$, show that

$$\mathbf{k}_{\theta} \cdot \hat{\xi}_{\perp 0} = 0.$$

{Note: $[\nabla \times (\hat{\xi}_{\perp} \times \mathbf{B})]_{\perp} = [(\mathbf{B} \cdot \nabla) \hat{\xi}_{\perp} - (\hat{\xi}_{\perp} \cdot \nabla) \mathbf{B}]_{\perp}$ } .

15 minutes

(II.2) Subsequently minimizing δW w.r.t. $\hat{\xi}_{\perp 1}$, show that

$$\overline{(i \mathbf{k}_{\theta} \cdot \hat{\xi}_{\perp 1} + \nabla \cdot \hat{\xi}_{\perp 0})} \equiv \overline{(\nabla \cdot \hat{\xi}_{\perp})_1} = -2 \overline{\mathbf{k} \cdot \hat{\xi}_{\perp 0}}^{-(1+\beta/2)}. \quad \underline{\text{END}}$$

[The final δW can then be expressed in terms of $\hat{\xi}_{\perp 0}$ and is independent of m .]

DEPARTMENT OF ASTROPHYSICAL SCIENCES

PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION

MAY 15, 1990

PART II

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- Follow the directions for each section carefully. NOTE: Sections IV (do either 4B-1 or 4B-2) and V (do either 5B-1 or 5B-2) have a choice of problems.
 - The total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to the allotted time.
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Elementary/General Plasma Physics

#1B - 20 minutes

Two identical spherical satellites (metal clad) with each surface area, S , are tied by a metallic rod of length, ℓ , the rod being clad by an insulator. Satellites are in a space plasma. The plasma has density, n , temperature, $kT = kT_e = kT_i$, average mass of ions, M and singly ionized. If two satellites are maintained at the potential difference, V (by a battery on board the satellite(s)):

1. Estimate the amount of an electric current that flows in the metal rod.
2. If the rod is perpendicular to the earth magnetic field, B , estimate the accelerating force of the satellite pair. Assume that the radius of each satellite is much smaller than ℓ .

#2B - 60 minutes

Starting from the ideal MHD equations, derive the equations for propagation of small amplitude waves with frequencies well below the ion cyclotron frequency ($\omega \ll \Omega_i = eB/M_i c$) in a plasma with constant density and straight, uniform field \vec{B}_0 . How many different types of waves exist? Derive their dispersion relations. The ideal MHD equations are approximations to a more complex real physics. What modifications must be made to the MHD equations so they are valid for $\Omega_i \lesssim \omega \ll \Omega_e$? For fusion plasmas such as tokamaks, which of the ideal MHD equations is the least satisfactory and why?

#3B

Kinetic Theory II (40 points)

A superthermal distribution of electrons is trapped initially in a mirror machine. In considering the slowing down and scattering of the superthermal electrons, assume that the background is a neutral electron-ion plasma. The evolution of the superthermal electron distribution function, f , given at the mirror symmetry plane, is described rather well for $t > 0$ by the following equation:

$$\frac{\partial f}{\partial t} = \Gamma \left[\frac{1}{v^2} \frac{\partial}{\partial v} f + \frac{1 + Z_i}{2v^3} \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial}{\partial \mu} f \right], \quad \text{for } -\mu_B < \mu < \mu_B \quad (*)$$

where v is the electron speed, $\mu = v_{\parallel}/v$ is the velocity pitch angle, and $\pm\mu_B$ are the pitch angles that bound the trapped region of electrons. For $\mu < -\mu_B$ or $\mu > \mu_B$, the electrons are considered to be lost instantaneously.

- (a) (10 pts) Interpret each of the terms in Eq.(*). What approximations were used in obtaining this equation from the Landau collision integral? In what region of velocity space does the equation fail to represent adequately the evolution of the distribution and what terms are neglected?
- (b) (10 pts) What are suitable boundary conditions for posing this equation? Consider in your answer that a numerical computation is anticipated, on a grid for which a maximum electron energy can be accommodated
- (c) (20pts) Based on the properties of Eq.(*) with suitable boundary conditions, which of the following statements are true? (You need not provide any proof, but a very brief explanation is appreciated.)
 - (i) The nonnegative nature of $f(\mu, v, t)$ is preserved.
 - (ii) The initial maximum of $f(\mu, v, t)$ is never exceeded.
 - (iii) Both the trapped electron density and trapped electron energy decrease monotonically in time.
 - (iv) The trapped particle entropy can both increase in time or decrease in time, but eventually it must be less than initially.
 - (v) For $n \geq 0$, the initial maximum of $v^n f(\mu, v, t)$ is never exceeded; for $n < 0$, it might be exceeded.

Section IV (do either #4B-1 or #4B-2)

EXPERIMENTAL METHODS

#4B-1

DENSITY INTERFEROMETRY (60 MIN).

1. (20 min.) Derive a relation between phase shift and plasma density from the dispersion relation for ordinary mode electromagnetic waves.
2. (10min.) What has one to do in order to get plasma density profiles?
3. (10min.) What is a Mach Zehnder and what a Michelson interferometer? Make two small sketches.
4. (20 min.) Provide a short explanation and description (microwave generator, beam optical components, detectors, signal processing) either for a Zebra stripe interferometer or a submillimeter interferometer like MIRI on TFTR.

EXPERIMENTAL METHODS

#4B-2

PLASMA RADIATION (60 minutes).

Consider the various types of plasma radiation losses listed below. Provide a short discussion with the following items. (a) Give a short description of each of the processes using small illustrative graphs. (b) Spell out the wavelength range in which the electromagnetic radiation is emitted. (c) Indicate which processes were predominant in the first tokamak plasmas (ST, PLT ohmic) that had a electron temperature $T_e \approx 1$ keV, and argue which processes you expect to be important in a tokamak fusion reactor with central electron temperatures of $T_e \approx 30$ keV.

- I. (12 min.) $\Delta n=0$ transitions. Discuss these individually for hydrogen, carbon, iron, and tungsten ions.
- II. (12 min.) $\Delta n=1$ transitions. Discuss these individually for hydrogen, carbon, iron, and tungsten ions.
- III. (12 min.) Bremsstrahlung. How does total bremsstrahlung loss scale with electron density, temperature, and Z_{eff} .
- IV. (12 min) Recombination Radiation. How can one experimentally distinguish between recombination radiation and bremsstrahlung.
- V. (12 min.) Synchrotron radiation. How does the synchrotron radiation from a single electron scale with magnetic field and electron energy? Discuss to what extent synchrotron radiation is black body radiation.

#5B-1

Quasilinear theory, etc. {60 minutes}

NOTE: More than half of this problem can be done without any knowledge of quasilinear theory whatsoever.

* * *

Assume that a plasma is adequately described by the following simple drift-wave equation for the electrostatic potential φ :

$$(1 - \nabla_{\perp}^2) \frac{\partial}{\partial t} \varphi(\mathbf{x}, t) + V_d \frac{\partial \varphi}{\partial y} = 0.$$

(a) [7 min.] What is the physical significance of each of the terms 1 , ∇_{\perp}^2 , and V_d ? Write down from memory, or derive if necessary, the expression for V_d . This formula will involve the density scale length

$$L_n \doteq \left(\frac{\partial}{\partial x} \ln n \right)^{-1} \doteq \kappa_n^{-1}, \quad (1)$$

where $n(x)$ is the background density.

* * *

Now assume that the plasma is bounded in x by two perfectly conducting plates at $x = 0$ and $x = L$. The plates are infinitely extended in y and z . The right-hand plate at $x = L$ is grounded. The potential on the left-hand plate at $x = 0$ obeys

$$\varphi(x=0, t) = \varphi_0 \cos(\omega_0 t),$$

where ω_0 is "high" in some sense.

(b) [6 min.] What does "high" mean? Be quantitative.

(c) [10 min.] Assuming that L_n is constant, determine the forced response $\varphi_{\mathbf{k}, \omega}(x)$, where

$$\varphi_{\mathbf{k}, \omega} = \int_{-\infty}^{\infty} dx dt e^{-i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \varphi(\mathbf{x}, t).$$

For future use, note that that this response vanishes under a time average.

* * *

Now suppose that the background density n has a *small, static, and random* component $\delta n(x)$:

$$n(x) = \langle n \rangle(x) + \delta n(x),$$

where $\langle n \rangle'(x)$ is now assumed to be constant. The correlation length of the fluctuations is assumed to be very short, so we take

$$C(x, x') \doteq \langle \delta n(x) \delta n(x') \rangle \approx \Delta \delta(x - x')$$

for some given constant Δ .

(d) [6 min.] In terms of a general $C(x, x')$, what is an approximate formula for the correlation function $K(x, x')$ of $\kappa_n(x)$ in the limit of small δn , where κ_n is defined by Eq. (1) above? That is,

$$K(x, x') \doteq \langle \delta \kappa_n(x) \delta \kappa_n(x') \rangle \approx ?$$

* * *

The ultimate goal is now to determine the contribution to the fluctuation spectrum caused by the random density modulations. In particular, we want to compute

$$S_{k,\omega}(x, x') \doteq \langle \delta \varphi_{k,\omega}(x) \delta \varphi_{k,\omega}^*(x') \rangle$$

for fixed x and x' . In general, this is a very difficult calculation. You will be asked to discuss only parts of it.

The exact equation for φ can be written in a canonical form that displays the random density explicitly. We write

$$\mathcal{L}_0 \varphi + \delta \mathcal{L} \varphi = 0,$$

where both \mathcal{L}_0 and $\delta \mathcal{L}$ are linear operators; $\delta \mathcal{L}$ is random while \mathcal{L}_0 is not.

(e) [6 min.] What are \mathcal{L}_0 and $\delta \mathcal{L}$ for this problem?

* * *

From now on, let us use the notation $\langle \dots \rangle$ to indicate an average over both time and an ensemble of random spatial modulations. Since $\delta n(x)$ is independent of time, we can conclude that $\langle \varphi \rangle = 0$.

(f) [6 min.] Construct an exact (but unclosed) equation for $S_{k,\omega}(x, x')$. Draw an analogy to the BBGKY hierarchy of classical kinetic theory.

(e) [19 min.] Using your knowledge of quasilinear theory and of truncations of the BBGKY hierarchy, discuss the problem of finding a closed (but approximate) equation for $S_{\mathbf{k},\omega}$; derive such an equation if possible. Discuss your assumptions as much as possible, and draw analogies to other similar problems you know how to solve. It is adequate to derive this equation in the quasilinear approximation; *you are not required to solve the equation*. Your equation should involve (1) either $C(x, x')$ or $K(x, x')$, and (2) the zeroth-order Green's function $G_{\mathbf{k},\omega}(x, x')$ for the fluctuations, which obeys

$$-\frac{d^2}{dx^2} G_{\mathbf{k},\omega}(x, x') + V_{\mathbf{k},\omega} G_{\mathbf{k},\omega}(x, x') = \delta(x - x'),$$

where V is the appropriate "potential" for this problem.

NOTE: You are *not* required to compute G explicitly.

If you have no idea how to do this part of the problem, you will earn a few points if you determine the explicit formula for G . However, do not spend more than about 5 minutes on that calculation. *Do not do this if you have successfully derived the formula for S or given the general discussion requested.*

#5B-2

Neoclassical Theory

60 minutes

Describe and give a physics argument which estimates, in a large aspect ratio circular tokamak,

a) bootstrap current

b) Ware pinch.