DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 17, 1993

9 a.m. - 1 p.m.

- Answer all problems.

- The exam has been designed to require 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem ____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Problems for Part I, May 17, 1993

1)	Applied Math	20 minutes
2)	Applied Math	25 minutes
3)	Kinetic Theory	60 minutes
4)	Waves Quickies	15 minutes
5)	Waves	40 minutes
6)	Kinetic Quickie	10 minutes
7)	Computational Methods quickie	10 minutes
	Total	180 minutes

Question # 1

Applied Math (20 minutes)

The incomplete gamma function is defined by

$$\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1}\, dt.$$

Find the first two terms in the asymptotic expansion of $\Gamma(a,x)$ for large x.

Question # 2

Applied Math (25 minutes)

Parabolic cylinder functions satisfy the equation

$$\frac{d^2y}{dx^2} - (\frac{1}{4}x^2 + a)y = 0.$$

(a) (5 min) What is the nature of the point at ∞? (Ordinary point, regular singular point, or irregular singular point?) Justify.
(b) (10 min) Find the controlling factors of y for $x \to \infty$.
(c) (10 min) Find the full leading order behavior of the dominant solution for $x \to \infty$.

Question # 3

Kinetic theory 60 minutes

Consider the scattering of laser light from an unmagnetized, weakly coupled, thermal-equilibrium plasma. It is asserted that in a certain regime of frequencies and wavenumbers the scattered power will exhibit a sharp spike near the plasma frequency. In this problem you are asked to examine the properties of that spike with the aid of a simple model.

Assume that the ions can be modeled as a randomly distributed collection of discrete, *stationary* scattering centers. Instead of using the Vlasov equation to describe the dielectric properties of the electrons, let the electrons obey the simple fluid model

$$\frac{\partial}{\partial t}n = -\nabla\cdot(n\boldsymbol{u}),$$

$$mn\frac{\partial}{\partial t}\boldsymbol{u} = nq\boldsymbol{E} - \nabla P - \nu\boldsymbol{u},$$

$$\frac{\partial}{\partial t}T = 0.$$

Here ν is supposed to model both the electron Landau damping as well as a frictional drag on the electrons due to the ions.

(a) [**20 min.**] Derive a formula for the electron density perturbation $\delta n^{\mathrm{ind}}(\boldsymbol{k},\omega)$ induced by a moving test charge of charge Q and velocity $\boldsymbol{V}$ inserted into the plasma.

(b) [**15 min.**] The scattered power is equal to $AC(\Delta\boldsymbol{k},\Delta\omega)$, where A is a known constant, Δ denotes the difference between the outgoing and incident radiation, and $C(\boldsymbol{k},\omega)$ is the equilibrium fluctuation spectrum of the electron density:

$$C(\boldsymbol{k},\omega) = \langle\delta n_e\,\delta n_e\rangle(\boldsymbol{k},\omega).$$

Find a formula for $C(\boldsymbol{k},\omega)$. Explain and justify all steps.

(c) [**10 min.**] Under what circumstances do you expect to see a well-defined line near the plasma frequency? Argue that your formula from part (b) is consistent with your answer.

(d) [**5 min.**] In a normalization such that $A = 1$, what procedure or formula would you use to determine the total scattered power contained in the plasma line(s)? State the steps you would take as explicitly as possible, but **DO NOT** actually evaluate whatever formulas you write down.

(e) [**5 min.**] Still assuming that $A = 1$, how do the height and width of the line scale with physical parameters? You may ignore numerical coefficients, so don't bother with a detailed algebraic calculation.

(f) [**5 min.**] Derive a criterion for $\Delta\boldsymbol{k}$ such that the width of the line is dominated by collisional drag instead of Landau damping.

Note: Parts (c)–(f) are mostly independent of each other. Each can be answered by using physical intuition or general principles; they don't necessarily require the detailed algebraic results from parts (a) and (b).

WAVES - Question # 4 15 minutes

Wave Quickies

(5 minutes) Given a wave dispersion relation, say $D(\omega, k)= 0$, one can solve for a temporal or spatial damping (ω_i and k_i, respectively) knowing the imaginary part, $D_i(\omega_r, k_r)$. Describe how ω_i and k_i are related.

(5 minutes) What are the assumptions that underlie quasilinear theory?

(5 minutes) What is the relation between the validity of quasilinear theory and wave-induced stochastic particle motion?

Question # 5

Waves (40 minutes)

You may recall that the equation of electrostatic waves in a one-dimensional unmagnetized Maxwellian plasma can be given as,

$$k^2 = \frac{1}{2} \sum_s \frac{1}{\lambda_{ds}^2} Z_0'\left(\zeta^{(s)}\right),$$

$$\frac{1}{\lambda_{ds}^2} = \frac{4\pi n_s q_s^2}{\kappa T_s},$$

$$Z_0'(\zeta^{(s)}) = \frac{d}{d\zeta} Z_0(\zeta^{(s)}) = -2\left[1 + \zeta^{(s)} Z_0(\zeta^{(s)})\right],$$

$$\zeta^{(s)} = \frac{\omega}{k\, v_{Ts}} \quad \textit{where} \quad v_{Ts} = \sqrt{\frac{2\kappa T_s}{m_s}}$$

You may also use the following approximate forms in the respective limits:

$$Z_0(\zeta^{(s)}) \approx -2\,\zeta^{(s)} + i\sqrt{\pi} \quad \textit{for} \quad \zeta^{(S)} << 1,$$

$$Z_0(\zeta^{(s)}) \approx -\frac{1}{\zeta^{(S)}} - \frac{1}{2\zeta^{(s)3}} + i\sqrt{\pi}\, e^{-\zeta^{(s)2}} \quad \textit{for} \quad \zeta^{(s)} >> 1,$$

The following questions are related to the ion acoustic waves propagating in a single ion species non-drifting Maxwellian unmagnetized plasma with given T_e and T_i.

(a) (15 minutes) Obtain the dispersion relation including Landau damping terms. Qualitatively describe the damping terms due to the ions and electrons.

(b) (5 minutes) Draw a dispersion relation curve(i.e. ω vs. k) (just qualitatively). Give the expression for the wave phase and group velocities (only for the linear regime).

c (~~b~~) (5 minutes) Comment on the damping of waves for $T_i \approx T_e$ and $T_i << T_e$.

d (~~c~~) (10 minutes) Derive an expression for the energy density and power flow. How does one account for the energy flux, $v_g W$, where W is the energy density. Remember that

$$W \approx \frac{1}{16\pi}\left[B^* \cdot B + E^* \cdot \frac{\partial}{\partial \omega}(\omega \varepsilon_h) \cdot E\right]$$

e (~~d~~) (5 minutes) What is the physical mechanism for this energy flow.

Question #6

Kinetic Quickie (10 minutes)

Consider a steady-state collisionless unmagnetized plasma in a one-dimensional electrostatic potential well. If the well has a single maximum and a single minimum, and the plasma is bounded by reflecting walls, which of the following statements is correct?

a. If the ion distribution function is isotropic at the bottom of the well, it is isotropic everywhere.

b. If the ion distribution function is isotropic at the top of the well, it is isotropic everywhere.

Give a reason for the correctness of one statement and a counterexample for the incorrectness of the other.

Question # 7 (10 minutes)

Quickie: Computational methods and applied math

Upwind differencing:

Consider the simple one-dimensional advection equation:

$$\frac{\partial a}{\partial t} + u \frac{\partial a}{\partial x} = 0 \quad (1)$$

where $a(x,t)$ is the unknown, and u is a constant. A possible finite difference method for solving this equation is given by

$$a_j^{n+1} = a_j^n - u \frac{\Delta t}{\Delta x} \left(a_j^n - a_{j-1}^n \right) \quad (2)$$

where $a_j^n \equiv a(j\Delta x, n\Delta t)$ **, with Δx being the cell size, and Δt being the time step.**

(i) What is the condition that the finite difference scheme (2) is numerically stable?

(ii) What is the physical interpretation of the stability condition in (i)?

(iii) What boundary conditions are needed to solve (2)?

(iv) What is the numerical truncation error caused by using the the finite difference equation (2) instead of the exact differential equation (1)?

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GENERAL EXAMINATION, PART II

MAY 18, 1993

9 a.m. - 1 p.m.

- Answer all problems. Problem 6 has a choice of A or B (answer one only).

- The exam has been designed to require 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

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Problems for Part II, May 18, 1993

1)	MHD Quickie	5 minutes
2)	General Plasma Physics	40 minutes
3)	MHD	40 minutes
4)	Kinetic Quickie	10 minutes
5)	General Plasma Physics	25 minutes
6A)	Experimental (Fabry-Perot interferometry)	60 minutes
or		
6B)	Experimental	60 minutes
	Total	180 minutes

Question #1

MHD Quickie (5 minutes)

If a plasma is stable according to the Mercier criterion, does this imply that it is stable to ballooning modes? What about the converse? Explain your reasoning.

Question #2

General plasma physics [40 minutes]

(a) [10 minutes] A low-β plasma is confined in a long cylindrical solenoid. Initially, the magnetic field is $\mathbf{B} = B_0\hat{z}$ and the plasma temperature is T_0. Between $t = 0$ and $t = \tau$, the magnetic field is increased to αB_0. Assume that τ satisfies $1/\Omega_i \ll \tau \ll 1/\nu_{ee}$, where Ω_i is the ion cyclotron frequency and ν_{ee} is the electron-electron collision frequency.

Suppose that an ion initially has velocity components (relative to the magnetic field) of $v_{\|} = v_{\|0}$, $v_{\perp} = v_{\perp 0}$, and that its guiding center is at radius $r = r_0$. What are the final values of $v_{\|}$, $v_{\perp}$, and r?

(b) [5 minutes] With the magnetic field held at αB_0, the plasma is now allowed to equilibrate . What is temperature of the plasma in this state?

(c) [10 minutes] The magnetic field is now reduced to B_0 once more (on the same time scale as the previous increase) and the plasma again allowed to equilibrate. What is the final temperature of the plasma?

(d) [15 minutes] Suppose that instead of changing the magnetic field as described in (a), the rate of change of the magnetic is extremely slow so that the plasma is always in equilibrium. Thus we take $\tau \gg 1/\nu_{ei}$ where ν_{ei} is the electron-ion energy equilibration rate. What is the temperature after the magnetic field becomes αB_0? What is the temperature after the magnetic field returns to B_0?

Question #3 40 minutes

MHD

For a low-β, nested circular surface tokamak, the uniform vertical field necessary to maintain an equilibrium is given by:

$$B_z = \frac{\mu_0 I_p}{4\pi R}\left\{\ell n\left(\frac{8R}{a}\right) - \frac{3}{2} + \beta_\theta + \frac{\ell_i}{2}\right\}$$

1. Give the appropriate definitions of β_θ and ℓ_i.

   Suppose the external vertical field is increased to new value B_z'. Further suppose that the plasma responds by evolving through a sequence of equilibria all the time obeying the ideal MHD constraint. Also each plasma element maintains constant entropy-per-unit-mass. The flux Ψ_c through the core (the "hole-in-the-doughnut") arising from the plasma current is

$$\Psi_c = \mu_o I_p R\left[\ell n\left(\frac{8R}{a}\right) - 2\right]$$

2. Compute the ratio B_z'/B_z needed to make the new major radius $R' = \frac{R}{2}$.
3. Compute the new values of the plasma current, minor radius, β_θ' and ℓ_i' in terms of their initial values.

Use the large-aspect-ratio expansion to simplify algebra whenever appropriate.

Question # 4   (10 minutes)

1. Kinetic Quickie

Suppose that an energetic beam of protons slows down through colliding with electrons in an inhomge-neous, neutral plasma. An individual proton can then be assumed to evolve according to

$$\frac{d\mathbf{v}}{dt} = -\nu(\mathbf{r})\mathbf{v},$$

where ν is a collision rate that depends on position only.

(a) (5 pts) Write a partial differential equation that describes the time-evolution of a distribution of protons, $f(\mathbf{r}, \mathbf{v}, t)$.

(b) (5 pts) Write down any quantities of physical interest that this partial differential equation conserves.

Question # 5 (25 minutes)

5. General Plasma

Consider a plasma in a uniform magnetic field. A test particle has speed u, cyclotron frequency Ω, Larmor radius ρ_L, and a 90 deg collision frequency ν_{coll}.

(a) (5 pts) What is the diffusion coefficient perpendicular to the magnetic field for a test particle in the limit $\nu_{\mathrm{coll}}/\Omega \gg 1$ and in the limit $\nu_{\mathrm{coll}}/\Omega \ll 1$?

(b) (5 pts) What is the diffusion coefficient parallel to the magnetic field for a test particle in the limit $\nu_{\mathrm{coll}}/\Omega \gg 1$ and in the limit $\nu_{\mathrm{coll}}/\Omega \ll 1$?

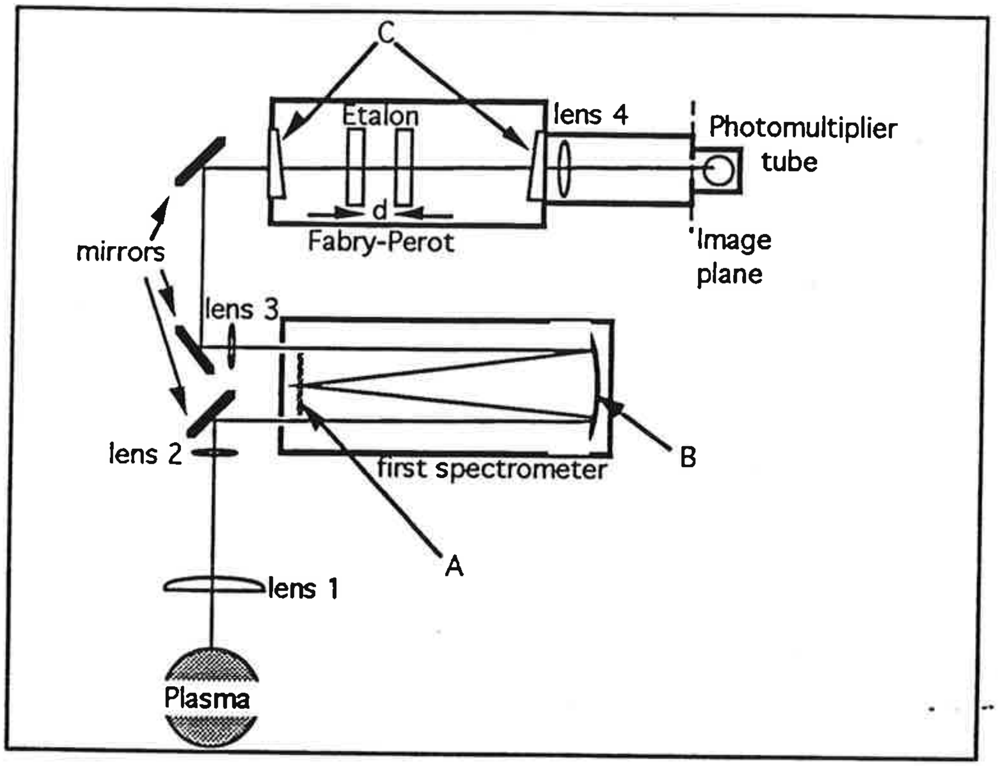
(c) (15 pts) Show that for diffusion across a strong magnetic field, like particle collisions and unlike particle collisions contribute very differently to the diffusion. What is "ambipolar diffusion" and what is the difference between intrinsic and extrinsic ambipolarity?

Question # 6A

Experimental Methods

Fabry-Perot Interferometry (60 minutes)

Fabry-Perot interferometry can be used to provide high-resolution analysis of spectral lines.



a) Consider the figure above. (20 minutes)

  i) Briefly explain the principles of Fabry Perot interferometry.
  ii) Explain the purposes of lenses 1-4.
  iii) What is the purpose of the first spectrometer?
  iv) Describe the functions of items A, B, and C.
  v) For a monochromatic plane wave input to the F-P, sketch the light intensity at the image plane.

b) Define the free spectral range (FSR). Express the FSR in terms of wavelength or frequency. The instrument resolution (bandwidth) is given by

$$\Delta\lambda_{BW} = \Delta\lambda_{FSR}/F,$$

where F is the finesse of the instrument. What parameters contribute to the finesse of a particular F-P spectrometer? What is a typical value for F? (20 minutes)

c) Describe, quantitatively where possible, what phenomena might contribute to the measured width of a spectral line. (20 minutes)

Question # 6B (Experimental) (60 minutes)

You have been put in charge of the Diagnostics Division for the Tokamak Physics Experiment (TPX). TPX is a new steady-state tokamak to be built at PPPL, whose mission is to develop new regimes of plasma operation with enhanced confinement and beta.

1. (10 minutes)
How would enhanced confinement and beta help improve a tokamak reactor? Why might you need them BOTH?

2. (15 minutes)
List at least 6 local plasma properties that you will need to measure as a function of minor radius in TPX, and explain why you need to measure them.

3. (25 minutes)
Name at least one plasma diagnostic technique that you can use to measure each of the quantities, and give the underlying physical mechanisms used for each. Be as specific as possible (e.g., if you are going to scatter photons from the plasma - what is the scattering cross-section?).

4. (10 minutes)
Discuss any difficulties you see with making these measurements in a steady-state device (with a pulse length >= 15 minutes).