DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 13, 1996

9 a.m. - 1 p.m.

- Answer all problems.

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

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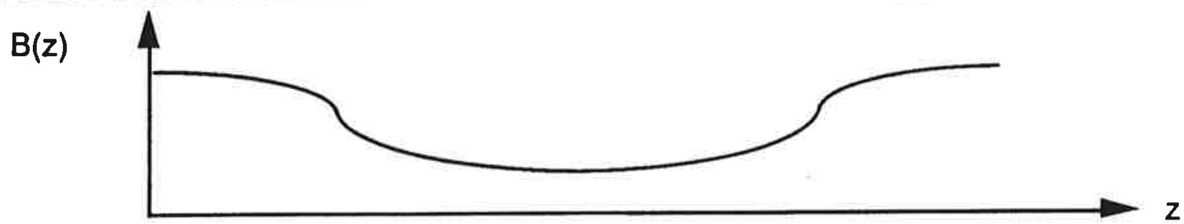
Problems for Part I, May 13, 1996

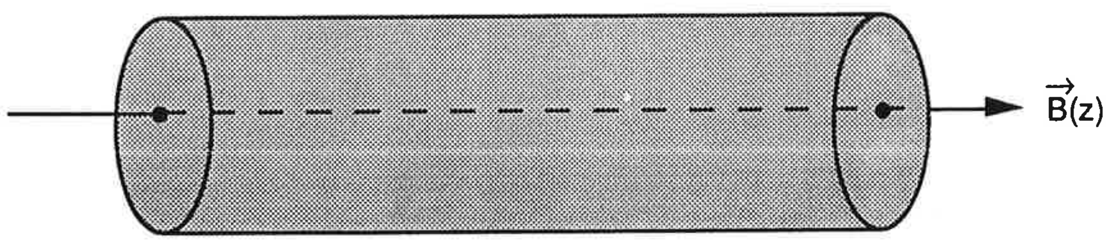
1)	Waves	15 points
2)	Irreversible Processes	45 points
3)	General Plasma Physics	40 points
4)	Experimental Methods	45 points
5)	Applied Math	30 points
6)	Computational Methods	20 points

Total - 195 points

Part I, Question 1:
Waves [15 mins.]

A cylindrical plasma column of uniform density is immersed in a poloidally symmetric, axial field that varies slowly in the z direction as shown in the figure.





Assuming that the plasma is cold and consists of only electrons and one ion species, the dispersion relation for purely parallel propagating waves at any axial location can be written approximately as

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = \frac{\omega^2 \pm \omega\Omega_e + \Omega_i\Omega_e - \omega_{\mathrm{pe}}^2}{(\omega \pm \Omega_i)(\omega \pm \Omega_e)}, \tag{1}$$

where

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = \begin{cases} R & \text{for } (+), \\ L & \text{for } (-), \end{cases} \tag{2}$$

and

$$\omega^2 = \omega_{\mathrm{pe}}^2 \quad \text{for } P = 0. \tag{3}$$

Note that $\Omega_s \stackrel{\text{def}}{=} q_s B/m_s c$ and $\omega_{\mathrm{ps}}^2 \stackrel{\text{def}}{=} 4\pi n_s q_s^2/m_s$, where q_s is the signed charge of the species and m_s is the mass of the species.

Suppose you want to use an antenna to launch purely propagating waves in this plasma that will heat ions.

(a) **[5 mins.]** On which branch of the dispersion relation would you want to launch waves and why?

(b) **[5 mins.]** Indicate on the diagram where would be an appropriate place to locate an antenna and indicate where ion heating might occur for your choice. Briefly justify your answer.

(c) **[5 mins.]** What condition must the density satisfy in order to insure that the launched waves will propagate?

Part I, Question 2:
Irreversible processes [45 mins.]

Consider a mirror machine consisting of a long, straight solenoid with magnetic field B_0 and with magnetic mirrors at either end with peak magnetic field B_1 ($B_1 > B_0$).

(a) [10 mins.] Derive a condition for particles to be confined in such a device, assuming that collisions and electric fields can be ignored. State any other assumptions you need to make.

(b) [5 mins.] Now consider the electrons in such a device. Under what condition(s) can the collision operator for the electrons be approximated by the Lorentz collision operator,

$$\frac{\partial f}{\partial t} = \left(\frac{Z_i n_e e^4 \log \Lambda}{8\pi \epsilon_0^2 m_e^2}\right) \frac{1}{v^3} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial f}{\partial \theta}, \tag{1}$$

where $\cos\theta \stackrel{\text{def}}{=} \widehat{\boldsymbol{v}} \cdot \widehat{\boldsymbol{B}}$?

(c) [10 mins.] Supposing that electrons of energy E_0 are injected into this device (e.g., by means of a neutral beam) with $v_{\|} = 0$ at a rate of S electrons per cubic meter per second. Rewrite Eq. (1) to include a suitable source term, and specify suitable boundary conditions to model the loss condition derived in part (b), assuming that the transit time across the machine is much less than the collision time.

(d) [15 mins.] Find the steady-state solution for f. (You may leave the solution in terms of an integral, if you want.)

(e) [5 mins.] Discuss other physical effects that will influence the loss of electrons in a real device.

Part I, Question 3:
General Plasma Physics [40 mins.]

Consider a perfectly conducting plasma that is immersed in a uniform z-directed magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$. Suppose a small, spherically symmetric explosion occurs that results in a symmetric fluid velocity

$$\boldsymbol{v}(\boldsymbol{r},t) = \frac{r_0^2}{r^2}c(t)\hat{\boldsymbol{r}}, \tag{1}$$

where $\hat{\boldsymbol{r}}$ is the unit vector in the radial direction and r_0 and $c(t)$ are given.

(a) [10 mins.] If the fluid density $n(\boldsymbol{r},t{=}0) = n_0$ (a constant), show that

$$n(\boldsymbol{r},t) = \begin{cases} n_0 & \text{if } r^3 > 3r_0^2 \int^t dt'\, c(t'), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

[*Hint:* You can either use the method of characteristics or argue the answer by following a point in the fluid.]

(b) [5 mins.] What equation governs the time evolution of the magnetic field?

(c) [5 mins.] Consider two particle trajectories, $\boldsymbol{r}_1(t) = (r_1(t), z{=}0)$ and $\boldsymbol{r}_2(t) = (r_2(t), z{=}0)$. What can you say about the magnetic flux through the surface (annulus) at $z = 0$ between $r_1(t)$ and $r_2(t)$?

(d) [15 mins.] Calculate $B_z(z{=}0, r, t)$.

(e) [5 mins.] Sketch the magnetic field lines in the x–z plane at some time $t > 0$. Where is B_z most intense?

Part I, Question 4:
Experimental Methods [45 mins.]

You are using an x-ray crystal spectrometer to measure ion temperature in a tokamak.

(a) [8 mins.] Discuss the basic principles of the measurement technique, and the general outlines of the technology used.

(b) [8 mins.] What atomic physics issues do you need to be cautious about?

(c) [8 mins.] Under what circumstances could the true temperature of the impurity species you measure be unequal to the true temperature of the bulk ions? Estimate this effect.

(d) [13 mins.] Estimate the effects of drift-wave turbulence on the accuracy of the ion temperature measurement.

(e) [8 mins.] Estimate the effects of a typical Mirnov oscillation on the ion temperature measurement.

Part I, Question 5:
Applied Math [30 mins.]

When two different waves are simultaneously present in a plasma, one of the waves can produce a modulation of the natural frequency of the other. To model this situation, consider the equation

$$\frac{d^2a}{dt^2} + \omega^2(t)a = 0. \tag{1}$$

Take

$$\omega^2(t) = \tfrac{1}{4} + \epsilon(\alpha + 2\cos t), \tag{2}$$

where ϵ is assumed to be small and α is a parameter. Obtain an asymptotic solution of the equation (to lowest order in ϵ) valid for $t = \mathcal{O}(1/\epsilon)$. For what values of α are there exponentially growing solutions? This is an example of a "parametric instability."

Here are some trigonometric identities that you may (or may not) find useful for this problem:

$$\sin(\theta_1)\sin(\theta_2) = \tfrac{1}{2}\cos(\theta_1 - \theta_2) - \tfrac{1}{2}\cos(\theta_1 + \theta_2), \tag{3a}$$
$$\cos(\theta_1)\cos(\theta_2) = \tfrac{1}{2}\cos(\theta_1 - \theta_2) + \tfrac{1}{2}\cos(\theta_1 + \theta_2), \tag{3b}$$
$$\sin(\theta_1)\cos(\theta_2) = \tfrac{1}{2}\sin(\theta_1 - \theta_2) + \tfrac{1}{2}\sin(\theta_1 + \theta_2). \tag{3c}$$

Part I, Question 6:
Computational Methods [20 mins.]

Consider the one-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \tag{1}$$

where $T(x,t)$ is the temperature, x is the distance, t is the time, and D is the (constant) diffusivity.

In the finite-difference method, the continuous temperature is replaced by the discrete array

$$T_j^n \stackrel{\text{def}}{=} T(x_j, t^n), \qquad x_j \stackrel{\text{def}}{=} j\Delta x, \quad t^n \stackrel{\text{def}}{=} n\Delta t, \tag{2}$$

where Δx and Δt are fixed space and time increments and j and n are integers. Consider the time-advancement scheme defined away from the boundary points by

$$T_j^{n+1} = T_j^n + S\left[\sigma\left(T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}\right) + (1-\sigma)\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right)\right], \tag{3}$$

where

$$S \stackrel{\text{def}}{=} \frac{D\Delta t}{(\Delta x)^2} \tag{4}$$

and σ $(0 < \sigma < 1)$ is a numerical parameter without direct physical significance.

(a) [15 mins.] For a given spatial increment Δx, a given value of σ, and a given diffusivity D, what is the condition on the time step Δt so that Eq. (3) is numerically stable?

(b) [5 mins.] What would be the advantage of using a value of $\sigma = 1/2$? Explain.

DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 14, 1996

9 a.m. - 1 p.m.

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<u>Problems for Part II, May 14, 1996</u>

1)	General Plasma Physics	5 points
2)	MHD	40 points
3)	Neoclassical Theory	30 points
4)	Waves	40 points
5)	Experimental Methods	30 points
6)	Fundamental Plasma Physics	40 points

<u>Total - 185 points</u>

Part II, Question 1:
General Plasma Physics [5 mins.]

- What is the "ignition point" of a fusion reactor.
- Should "magnetic fusion" reactors be designed to operate at the ignition point—or at some other point or points? Specify.

Part II, Question 2:
MHD [40 mins.]

Let a magnetic field be described in cylindrical coordinates (R, Θ, Z), such that the toroidal field is B_Θ and the poloidal field is $B_p = B_R\widehat{\boldsymbol{R}} + B_Z\widehat{\boldsymbol{Z}}$. Assume that the poloidal field vanishes, $B_p = 0$, and that the toroidal field B_Θ and the pressure p are *axisymmetric*—i.e., functions of R and Z alone. Show that for an axisymmetric equilibrium p is a function of R alone—i.e., $p = p(R)$. From this, show that if p is zero on a toroidal boundary then there is no confined magnetostatic equilibrium in the absence of a poloidal field.

To do this:

(a) [5 mins.] Derive the poloidal current j_p in terms of $F \stackrel{\text{def}}{=} RB_\Theta$.

(b) [10 mins.] Write down the R and Z components of the force equation.

(c) [5 mins.] Show that F is a function of p alone.

(d) [5 mins.] Show that, unless p is a constant everywhere (which is actually what we are trying to show), then

$$F\frac{dF}{dp} = -4\pi R^2. \tag{1}$$

(e) [10 mins.] From this prove that, unless $d^2(F^2)/dp^2 = 0$, p is a function of R alone. [*Comment:* $d^2(F^2)/dp^2$ actually can't vanish unless p is a constant, but you are not expected to show this.]

(f) [5 mins.] Finally, from the boundary condition $p = 0$ on the boundary, show that p vanishes everywhere.

Part II, Question 3:
Neoclassical Theory [30 mins.]

(a) **[4 mins.]** What is meant by a "banana orbit" of a trapped particle in a low-collisionality tokamak? Sketch it, showing clearly the three-dimensional nature of the orbit.

(b) **[7 mins.]** For a conventional tokamak with a circular cross section, use the constants of a particle's motion to calculate the width of the "fattest" banana as a multiple of the particle's Larmor radius, obtaining if possible the correct numerical coefficient.

(c) **[5 mins.]** By using a simple random-walk argument, write down a rough estimate for the cross-field particle diffusivity in the lowest-collisionality neoclassical regime of a tokamak.

(d) **[7 mins.]** Now add collisions, so that the plasma is in the "plateau" neoclassical regime. Again using a simple random-walk argument, obtain a rough estimate for the particle diffusivity in this regime.

(e) **[7 mins.]** For the case of electrons, explain for both neoclassical regimes whether your collision frequency refers to the electron–ion or the electron–electron collision frequency. Would your formulas be different in an imaginary case in which the electron–electron collisional frequency were much larger than the electron–ion collision frequency?

Part II, Question 4: Waves [40 mins.]

The electrostatic wave equation for small-amplitude modes in a uniform fully-ionized magnetized non-relativistic plasma is, for $k_y \equiv 0$,

$$0 = k_x^2 + k_z^2 + \sum_s \sum_n \omega_{ps}^2 \int_{-\infty}^{\infty} dv_z \int_0^{\infty} 2\pi v_\perp \, dv_\perp \, \frac{J_n^2(z_s)}{\omega - k_z v_z - n\Omega_s} \left[k_x \left(\frac{n}{z_s}\right) \frac{\partial f_{0s}}{\partial v_\perp} + k_z \frac{\partial f_{0s}}{\partial v_z} \right], \quad (1)$$

where $\Omega_s \stackrel{\text{def}}{=} q_s B_0 / m_s c$, $z_s \stackrel{\text{def}}{=} k_x v_\perp / \Omega_s$, and $f_{0s} = f_{0s}(v_\perp, v_z)$.

(a) [5 mins.] What is the physical significance of the Bessel function, $J_n(z_s)$?

(b) [5 mins.] What is the physical significance of the resonant denominator? How is the singularity resolved mathematically?

(c) [20 mins.] Assume that $f_{0s}(v_\perp, v_z)$ is a bi-Maxwellian,

$$f_{0s}(v_\perp, v_z) = \frac{1}{\pi w_\perp^2} e^{-v_\perp^2/w_\perp^2} \frac{1}{\sqrt{\pi} w_\parallel} e^{-v_z^2/w_\parallel^2}. \quad (2)$$

Show that the dispersion relation may be written as

$$0 = k_x^2 \epsilon_{xx} + k_z^2 \epsilon_{zz}, \quad (3)$$

where

$$\epsilon_{xx} = 1 + \sum_s \sum_n \omega_{ps}^2 \frac{e^{-\lambda} I_n(\lambda) n}{\lambda k_z w_\parallel \Omega} Z(\zeta_n), \quad (4a)$$

$$\epsilon_{zz} = 1 + \sum_s \sum_n \omega_{ps}^2 \frac{e^{-\lambda} I_n(\lambda)}{w_\parallel^2 k_z^2} 2[1 + \zeta_n Z(\zeta_n)], \quad (4b)$$

where

$$\zeta_n \stackrel{\text{def}}{=} \frac{\omega - n\Omega}{k_z w_\parallel}. \quad (5)$$

Assume $k_z > 0$.

Hints: The following relations may be useful:

$$\frac{1}{\pi w_\perp^2} \int_0^{\infty} 2\pi v_\perp \, dv_\perp \, J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{-v_\perp^2/w_\perp^2} = e^{-\lambda} I_n(\lambda), \quad (6)$$

where

$$\lambda \stackrel{\text{def}}{=} \frac{k_\perp^2 w_\perp^2}{2\Omega^2}, \qquad w_\perp^2 \stackrel{\text{def}}{=} \frac{2kT_\perp}{m}. \quad (7)$$

Also,

$$Z(\zeta) \stackrel{\text{def}}{=} \frac{1}{\sqrt{\pi}} \int_0^{\infty} dz \, \frac{e^{-z^2}}{z - \zeta} \qquad (\operatorname{Im} \zeta > 0). \quad (8)$$

(d) [10 mins.] Use the result from part (c) to find the dispersion relation for electron Bernstein waves, namely $k_\parallel = 0$ and $\omega > |\Omega_e|$.

Part II, Question 5:
Experimental Methods [30 mins.]

This problem concerns the sheath around probes.

(a) **[10 mins.] Electrically floating probe:** Assume a collisionless plasma with Maxwellian electrons at a temperature T_e. Derive the nonlinear differential equation relating the potential distribution to the electron and ion densities.

(b) **[20 mins.] Strongly (negatively) biased probe:** Derive the thickness of the "high-voltage" sheath under the assumptions that

1. the probe is biased to repel all electrons;
2. ion impact does not cause secondary electron emissions; and
3. ions arrive at the sheath with the "Bohm" speed u_B.

Part II, Question 6:
Fundamental Plasma Physics [40 mins.]

Plasma echoes are one of the paradigm nonlinear problems that illustrate important properties of plasma kinetic theory. In this problem, you will investigate a plasma echo in the context of the drift-kinetic equation.

Consider a plasma in a uniform magnetic field $\boldsymbol{B} = B\,\hat{\boldsymbol{z}}$. The collisionless drift-kinetic equation for $f = f(\boldsymbol{x}, v, t)$ is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} f + \frac{e}{m}E_z\frac{\partial f}{\partial v} = 0, \qquad (1)$$

where v is the particle velocity in the z direction (the perpendicular particle gyromotion has been averaged out), $\boldsymbol{V}_E$ is the $\boldsymbol{E}\times\boldsymbol{B}$ drift, and we will assume that the electrostatic approximation holds, $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$.

(a) [5 mins.] Conservation properties: Show that Eq. (1) conserves any quantity I of the form

$$I \stackrel{\text{def}}{=} \int d\boldsymbol{x} \int dv\, g(f(\boldsymbol{x}, v, t)), \qquad (2)$$

where g is any function of f. [Thus, in particular, the equation conserves the entropy (where $g = f \ln f$).]

(b) [7 mins.] Linear response: For the rest of this problem, we will ignore E_z and just focus on the effects of a *specified* $\boldsymbol{E}\times\boldsymbol{B}$ flow.

Calculate the linear response, namely the solution to

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial z} = -\boldsymbol{V}_E \cdot \boldsymbol{\nabla} f_0, \qquad (3)$$

under the following assumptions:

1. For simplicity, consider a single Fourier component of $\boldsymbol{V}_E$ with $k_x = 0$, which is pulsed on for an instant at time $t = 0$. Thus, $\boldsymbol{V}_E$ is (the real part of)

$$\boldsymbol{V}_E = \hat{\boldsymbol{x}}\, v_0 e^{i(k_y y + k_z z)}\delta(t). \qquad (4)$$

2. Assume that f_0 is a Maxwellian with a weak density gradient in the x direction, so $f_0 \approx (1 - x/L_n) f_{\mathrm{M}}(v)$. I.e., we can assume that we are looking at small scales compared to the density scale length, $x \ll L_n$, so that $\boldsymbol{\nabla} f_0 = -\hat{\boldsymbol{x}} f_{\mathrm{M}}(v)/L_n$ is independent of position.

(c) [8 mins.] Phase mixing:

- At any fixed position $\boldsymbol{x}$ and fixed velocity v, you should have found in part (b) that f_1 is an oscillatory function of time for $t > 0$ and does not decay. Sketch a plot of the real part of f_1 vs. v at several times.

- Based on these plots, consider the temporal behavior of the density

$$n_1(\boldsymbol{x}, t) \stackrel{\text{def}}{=} \int dv\, f_1. \tag{5}$$

Explain why the density will eventually decay in time even though f_1 does not. What is the characteristic time scale for this decay?

(d) [20 mins.] A plasma echo: Even after n_1 has decayed away due to phase mixing, there is still information stored in f_1. It is possible to extract this information by nonlinearly interacting it with a second electric-field pulse at a different wave number $\boldsymbol{k}'$. Assume that at a later time $t' \gg 0$ (t' sufficiently large that n_1 has decayed due to phase mixing) a second electric field is pulsed on, giving rise to an $\boldsymbol{E}\times\boldsymbol{B}$ flow now chosen to be in the y direction, with $k_y = 0$. This second $\boldsymbol{V}_E'$ pulse can thus be written as (the real part of)

$$\boldsymbol{V}_E' = \hat{\boldsymbol{y}}\, v_0' e^{i(k_x' x + k_z' z)} \delta(t - t'). \tag{6}$$

This second pulse produces no linear perturbation in f because $\partial f_0 / \partial y = 0$. However, it will produce a second-order nonlinear perturbation in f_2, due to the nonlinear interaction $\boldsymbol{V}_E' \cdot \boldsymbol{\nabla} f_1$, where f_1 was produced by the first pulse in $\boldsymbol{V}_E$. Thus f_2 could be found by solving the equation

$$\frac{\partial f_2}{\partial t} + v \frac{\partial f_2}{\partial z} = -\boldsymbol{V}_E' \cdot \boldsymbol{\nabla} f_1. \tag{7}$$

Note that f_2 will have Fourier components at $\boldsymbol{k}'' = \boldsymbol{k} \pm \boldsymbol{k}'$.

- Show that for one of these components, the resulting density $n_2 \stackrel{\text{def}}{=} \int dv\, f_2$ will "un-phase-mix," producing a density "echo" at a later time if $k_z' > k_z$.

- What is the time at which n_2 reaches its maximum value—i.e., what is the time of the echo?

[*Possible hint:* First show that the various Fourier components of f_2 can be written in the form

$$f_2 \propto f_{\mathrm{M}}(v) e^{\boldsymbol{k}'' \cdot \boldsymbol{x}} e^{is(t)v}, \tag{8}$$

where $s(t)$ is some time-dependent coefficient. Then find the time at which $s(t)$ vanishes.]