DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 19, 1997

9 a.m. - 1 p.m.

- Answer all problems, **except where choices are indicated**.

- The exam has been designed to require about 3 hours of work. However, the total time allotted is 4 hours.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem ____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, notes, etc.) except for an NRL formulary are permitted.

Problems for Part I, May 19, 1997

1)	Waves	40 points
2)	Neoclassical Theory	40 points
3)	Nonlinear Processes	35 points
4)	General Plasma Physics	20 points
5)	Experimental Methods	35 points
6)	Kinetic Theory	10 points
		Total - 180 points

Part I, Question 1:
Waves [40 points]

A pulsar in our galaxy is emitting narrow pulses of electromagnetic waves in the radio frequency regime. Assume that the interstellar magnetic field can be ignored and that the average interstellar electron density is about 0.03 cm^{-3}.

(A) **[5 points]** What is the electron plasma frequency in the interstellar medium?

(B) **[5 points]** What is the approximate dispersion relation for the transverse electromagnetic waves that we are detecting in the radiofrequency regime, i.e., $f \sim 200$ MHz?

(c) **[30 points]** We observe a time delay, $\Delta t = t_2 - t_1 = 0.001$ seconds, between the arrival of the pulse at frequencies given by $f_1 = 200$ MHz and $f_2 = f_1 - \Delta f = 183.4$ MHz, respectively. Derive a formula for the distance to the pulsar in terms of Δt, f_1, and f_2 (or f_1 and Δf) and then use it to estimate the distance.

Part I, Question 2:
Neoclassical Theory [40 points]

In an axisymmetric toroidal plasma, the conservation of canonical angular momentum is a very useful property which can be invoked to help estimate key neoclassical transport properties such as the inward particle pinch velocity ("Ware Pinch") and the banana excursion of trapped particles.

[5 pts.] (a) Express the conservation of canonical angular momentum, P_ζ, in terms of the poloidal flux function, ψ.

[10 pts.] (b) Show that $\partial\psi/\partial t = -\,\mathrm{v}\cdot\nabla\psi$ by assuming $v_\zeta \simeq v_\parallel$ in part (a).

[10 pts.] (c) Estimate the trapped-particle radial velocity by using Faraday's Law together with the result from part (b).

[5 pts.] (d) Using $\underline{B} = \nabla \times \underline{A}$, now express P_ζ in terms of B_θ.

[5 pts.] (e) Expand around r_0, the mean radius of a trapped-particle orbit, to express the result from part (d) in terms of the trapped-particle radial excursion,

$$\Lambda \equiv r - r_0 .$$

[5 pts.] (f) Taking $v_\zeta \simeq v_\parallel \simeq \varepsilon^{1/2} v$ (with $\varepsilon \equiv r/R_0$) and $B_\theta \simeq B_p$ (poloidal magnetic field), obtain an estimate for Λ in terms of the gyroradius.

Part I, Question 3:
Nonlinear Processes [35 points]

Plasmas support numerous linear waves. In this question, we explore some properties of these waves when weak nonlinear effects are included. For simplicity we treat a one-dimensional system. Consider three waves

$$E_j = \operatorname{Re} \mathcal{E}_j \exp(ik_j x - i\omega_j t),$$

for $j = 1, 2, 3$. We assume here that $\omega_j > 0$. If (k_j, ω_j) satisfies the linear plasma dispersion relation, $D(k_j, \omega_j) = 0$, then the complex amplitude $\mathcal{E}_j$ is a constant (assuming that the wave is not linearly damped). With the inclusion of nonlinear terms, these waves become coupled and $\mathcal{E}_j$ become a slowly varying function of space and time.

(a) [10 points] It can be shown that the coupling is strongest if the wavenumbers and frequencies satisfy the resonance conditions

$$k_1 = k_2 + k_3,$$

$$\omega_1 = \omega_2 + \omega_3.$$

By considering the plasma fluid equations, show *qualitatively* why this is so.

(b) [5 points] Considering an unmagnetized plasma, sketch the dispersion relation (ω vs k) for Langmuir waves and ion acoustic waves. Identify a possible triplet of waves satisfying the resonance conditions.

(c) [10 points] The equations for the evolution of $\mathcal{E}_j$ are most simply given in terms of the *action amplitude* $a_j \propto \mathcal{E}_j$ such that

$$n_j = \frac{W_j}{\omega_j} = p_j \left| a_j \right|^2,$$

where n_j is the action density, W_j is the wave energy density and $p_j = \pm 1$ is the sign of the wave energy. If we consider only temporal coupling (i.e., all the wave amplitudes are assumed to be spatially uniform), it is found that the evolution of a_j is given by

$$\frac{\partial}{\partial t} a_1 = p_1 K a_2 a_3,$$

$$\frac{\partial}{\partial t} a_2 = -p_2 K^* a_1 a_3^*,$$

$$\frac{\partial}{\partial t} a_3 = -p_3 K^* a_1 a_2^*.$$

Show that the total wave energy density is conserved, assuming that this is the sum of the individual wave energy densities.

(d) [10 points] Show that the action densities $(n_1 + n_2)$ and $(n_1 + n_3)$ are conserved and hence show that the power transfer between the any two waves will always be in the ratio of their frequencies.

Part I, Question 4:
General Plasma Physics [20 points]

(a) (10 points) Sketch the orbit of a particle trapped in the earth's magnetosphere, pointing out the 3 types of periodic motion. Use order-of-magnitude estimates to estimate the ratios of the 3 frequencies associated with these periodic motions. What is the main parameter which controls these ratios?

(b) (5 points) For a 1 keV proton at $R \approx 5R_E$, in the deeply trapped limit (remains near the equatorial plane), give an order of magnitude estimate of the time it takes to precess around the earth once. (The earth's radius $R_E \approx 6400$ km, the magnetic field at the earth's surface on the equator is ≈ 0.3 Gauss.) Show that the precession time is the same for 1 keV electrons.

(c) (5 points) Give an order-of-magnitude estimate of the collision frequency of 1 keV ions (at $n = 1/\ \mathrm{cm}^3$), and the number of times they will precess around the earth before being lost.

Part I, Question 5:
Experimental Methods [35 points]

Instructions:
Answer question # 1 and
either question # 2 or question # 3

1. [10 points]
Explain two ways how to measure plasma pressure.

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2. [25 points]
Design a configuration which shows how a baratron (neutral gas) pressure gauge might be used to measure plasma pressure. As a starting point to your answer consider the plasma sheath at a material boundary (one of the baratrons plates) and calculate the forces on the material due to the momentum flux and the electric field. Assume a plasma with cold ions and warm electrons.

OR

3. [25 points]
Show that the floating potential of a Langmuir probe in an unmagnetized plasma with $T_e >> T_i$ is given by

$$\phi = (T_e/2e)\,[\ln\,(2\pi\, Z_i m_e/m_i)\ -1]$$

Hint: The potential at the sheath/pre-sheath boundary is $\phi = -T_e/(2e)$. Discuss the physics that sets this potential, and derive the rough numerical value of ϕ in a one-liner.

Part I, Question 6:
Kinetic Theory [10 points]

For *classical* transport across a strong magnetic field, the thermal conductivity $\kappa_\perp$ is dominated by the ions, and the density diffusion coefficient $D_\perp$ is very much smaller than $\kappa_\perp$:

$$\frac{D_\perp}{\kappa_\perp} = \left(\frac{m_e}{m_i}\right)^p \tag{1}$$

for some power $p > 0$.

(a) [5 points] Determine p by making simple random-walk estimates for $\kappa_\perp$ and $D_\perp$.

(b) [5 points] Explain physically why it turns out that $D_\perp \ll \kappa_\perp$ rather than $D_\perp \sim \kappa_\perp$.

DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 20, 1997

9 a.m. - 1 p.m.

- Answer all problems, **except where choices are indicated.**

- The exam has been designed to require about 3 hours of work. However, the total time allotted is 4 hours.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

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Problems for Part II, May 20, 1997

1)	Waves	25 points
2)	Fusion Processes	20 points
3)	Magnetohydrodynamics	55 points
4)	Irreversible Processes	45 points
5)	Computational Methods/Applied Mathematics	45 points
		Total - 190 points

Part II, Question 1:
Waves [25 points]

$\boxed{\frac{d1}{dx} \approx n}$

1. [10 points] Hybrid Resonances.

Give a physical picture of the ion-ion hybrid resonance in terms of the ion drift responses. How does this physical picture differ for the upper hybrid resonance?

Hint: $S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}$ where the s labels the charge species.

2. [15 points] Ion Acoustic Waves and the Plasma Dispersion Function, Z.

Ion acoustic waves propagate in a non-drifting, Maxwellian, unmagnetized plasma with electron and ion temperatures T_e and T_i. Assume that the damping is weak.

(a) [9 points] Obtain the dispersion relation by expanding the Z-functions, retaining terms to leading non-vanishing order.

(b) [6 points] Comment on the dependence of the damping on T_i/T_e.

Hint: Recall that for a magnetized plasma, $K_{zz} = 1 + \sum_s \frac{2\omega_{ps}^2}{k^2 V_s^2} (1 + \xi_{s0} Z_0)$, where the s labels the charge species.

Part II, Question 2:
Fusion Processes [20 points]

1. (5 points)
What types of fusion-reactions in the Sun have been the main source of solar-heating on Earth?

2. (5 points)
What sorts of "man-made" fusion-reactions would be economically most appropriate on Earth?

3. (10 points)
What are the most significant differences, if any, between
(1) solar fusion-fuels
and
(2) economically appropriate terrestrial fusion-fuels?

Part II, Question 3:

Magnetohydrodynamics [55 points]

Instructions:

Answer either question #1 or question #2

1

1. (a) (10 points) Consider a pressureless plasma ($p = 0$) enclosed in a perfectly conducting wall, with $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$ at the wall. Show that the magnetic energy is conserved in ideal MHD. (When you discard terms, make sure it is clear why.)

   (b) (10 points) Obtain an expression for the decay of the energy in the presence of a small plasma resistivity. (Assume that the wall is still perfectly conducting.)

   (c) (5 points) The quantity

$$\int_V \mathbf{A} \cdot \mathbf{B} \, d^3x$$

   is called the magnetic helicity, where V denotes the volume occupied by the plasma. Show that under the above conditions (plasma bounded by a perfectly conducting wall, with $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$ at the boundary), the total helicity is invariant under a gauge transformation. (That is, the magnetic helicity is a well defined quantity in that case).

   (d) (10 points) Show that the total magnetic helicity in the plasma is a conserved quantity in ideal MHD.

   (e) (10 points) Obtain an expression for the decay of the magnetic helicity in the presence of a small plasma resistivity.

   (f) (10 points) In the presence of tearing modes, the effects of resistivity tend to be localized in the neighborhood of rational surfaces, where current sheets form. To estimate the effect of a tearing mode on the energy and magnetic helicity, assume that there is a localized current forming a sheet of width ϵ, with the magnitude of the current density in the sheet proportional to $1/\epsilon$. How does the decay of energy and magnetic helicity scale with ϵ for small ϵ? What can we conclude about the conservation of these quantities in the presence of tearing modes if the resistivity is small?

OR (see next page)

Part II, Question 3:

Magnetohydrodynamics [55 points]

Instructions:

Answer either question #1 or question #2

#2

2. Consider a solid infinitely conducting disk of mass M and radius a placed at the origin in the $z = 0$ plane in a uniform magnetized plasma and oriented normal to the uniform magnetic field $B_0\hat{\mathbf{z}}$. (r, θ, z are cylindrical coordinates.)

a) (15 points) Show that the equation for a linearized toroidal shear alfven wave propagating parallel to $\mathbf{B}$, i. e. an axisymmetric wave with only a toroidal velocity v_θ and a perturbed magnetic field B_θ, is

$$\frac{\partial^2 v_\theta}{\partial t^2} = v_A^2 \frac{\partial^2 B_\theta}{\partial z^2} \tag{0.1}$$

where v_A is the alfven speed. Also show that $\partial B_\theta / \partial t = B_0 \partial v_\theta / \partial z$.

b) (5 points) Show that v_θ and B_θ have a Dalembertian solution

$$v_\theta = v_A f(z - v_A t) g(r) \tag{0.2}$$

$$B_\theta = -B_0 f(z - v_A t) g(r) \tag{0.3}$$

for any functions f and g.

c) (10 points) Now, consider that the disk rotates with small angular velocity $\Omega(t)$. What is the solution for v_θ and B_θ in the wave that this rotation induces in the plasma for $z > 0$. (Use the condition at the face of the disk $z = 0, r < a$, that v_θ of the plasma is equal to v_θ of the disk.)

d) (10 points) Find the magnetic stress and the resultant torque on the disk from the linearized wave from the linearized part of Maxwell stress tensor, $\mathbf{T} = (B^2\mathbf{I}/8\pi - \mathbf{BB}/4\pi)$, evaluated at the disk where $\mathbf{I}$ is the unit dyadic.

e) (15 points) Use your result to show that if there is no other force on the disk then the slowing down of the disk is given by

$$\frac{d\Omega}{dt} = -\frac{bB_0^2 a^2}{4Mv_A}\Omega \tag{0.4}$$

where b is a numerical constant you should evaluate. Take as given that the moment of inertia of the disk is $Ma^2/2$, and that the rate of change of angular momentum of the disk is given by the torque on it.

(If you have difficulty with the first part you should take the results of it as given and do the rest of the problem.)

Part II, Question 4: Irreversible Processes [45 points]

This problem is concerned with fluctuations and transport in a gyrokinetic plasma.

A gyrokinetic plasma in the limit $T_i \to 0$ and in the electrostatic approximation consists of a collection of discrete gyrocenters described by the following gyrokinetic Klimontovich equation and associated gyrokinetic Poisson equation:

$$\frac{\partial}{\partial t}\widetilde{N}(\boldsymbol{R}, v_\parallel, \mu, t) + v_\parallel \frac{\partial \widetilde{N}}{\partial z} + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}\widetilde{N} + \frac{q}{m}E_\parallel \frac{\partial \widetilde{N}}{\partial v_\parallel} = 0, \tag{1a}$$

$$(\nabla^2 + \epsilon_\perp \nabla_\perp^2)\varphi = -4\pi\rho, \tag{1b}$$

where $\epsilon_\perp = \omega_{\mathrm{p}i}^2/\omega_{\mathrm{c}i}^2$ is the dielectric permittivity of the gyrokinetic vacuum. For this entire problem, we will consider

- **thermal equilibrium** (so there are no spatial gradients), and
- **weak coupling** (so the gyrocenter dynamics are dominated by parallel motions of the gyrocenters).

(a) [14 points] The gyrokinetic dielectric function $\mathcal{D}(\boldsymbol{k}, \omega)$ can be identified by calculating the total infinitesimal response to an external perturbing potential:

$$\varphi_{\boldsymbol{k},\omega} = \varphi_{\boldsymbol{k},\omega}^{\text{ind}} + \varphi_{\boldsymbol{k},\omega}^{\text{ext}} = \frac{\varphi_{\boldsymbol{k},\omega}^{\text{ext}}}{\mathcal{D}(\boldsymbol{k}, \omega)}. \tag{2}$$

Here ind means induced (internally) and ext means external. **Calculate the gyrokinetic dielectric function for a thermal-equilibrium (Maxwellian) plasma with no background gradients in the limit of weak coupling.**

Hint: You might find it easier to work directly with $E_\parallel = -ik_\parallel\varphi$ instead of φ.

(b) [6 points] **Very briefly, qualitatively describe the expected spatial behavior of the shielding cloud around a stationary gyrocenter.** In particular, identify the characteristic shielding lengths $L_\parallel$ and $L_\perp$ in the parallel and perpendicular directions, respectively. You may use the specific form of $\mathcal{D}(\boldsymbol{k}, \omega = 0)$ to help you, but it is not necessary (and there isn't enough time) to evaluate any integrals.

(c) [15 points] **Use the Test Particle Superposition Principle to calculate the Fourier transform of the two-point charge-density fluctuations due to the gyrocenter discreteness:** $\langle \delta\rho\, \delta\rho \rangle(\boldsymbol{k}, \omega) = ???$ **Briefly describe the basis of this principle.**

(Problem continues on next page.)

(d) [**10 points**] Suppose one extra test gyrocenter is placed into this thermal-equilibrium, statistically fluctuating plasma. **Using simple random-walk arguments** *(not detailed formal mathematics)*, **estimate the cross-field spatial diffusion coefficient $D_\perp$ for that test gyrocenter.** This will require you to estimate the total fluctuation level $\mathcal{E}$ and identify an appropriate autocorrelation time $\tau_{\rm ac}$. **Motivate whatever formula you use for $D_\perp$ as well as your choice for $\tau_{\rm ac}$.**

Hint: Recall that for an ordinary unmagnetized thermal-equilibrium plasma the wave-number fluctuation spectrum is

$$\frac{\langle \delta E^2 \rangle(\boldsymbol{k})}{8\pi} = \frac{\frac{1}{2}T}{1+(k\lambda_{\rm D})^2}. \tag{3}$$

Without worrying about possible divergences in the wave-number integral, the total fluctuation level for that spectrum is

$$\mathcal{E} = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \langle \delta E^2 \rangle(\boldsymbol{k}) \sim (8\pi)(\tfrac{1}{2}T)(k_{\rm D}^3) \tag{4}$$

or

$$\frac{\mathcal{E}}{8\pi\overline{n}T} \sim \epsilon_{\rm p} = 1/\overline{n}\lambda_{\rm D}^3. \tag{5}$$

Analogous arguments can be used to estimate $\mathcal{E}$ for the gyrokinetic plasma. You don't have time to calculate the formula analogous to Eq. (3); just make an intelligent guess about its behavior.

Part II, Question 5:

Computational Methods /Applied Mathematics [45 points]

Instructions:

Answer either question #1 or question #2

1. [45 points]

Consider the hyperdiffusion equation

$$\frac{\partial U}{\partial t} = -\lambda \frac{\partial^4 U}{\partial x^4}$$

where $U(x,t)$ is the unknown, λ is the (constant) hyperdiffusion coefficient, and the time t and the position x are the independent variables. Assume that $U(x,0)$ is given on the spatial domain $0 < x < L$.

(1) (10 pts) Write down an *explicit* finite difference equation, valid for any interior point, to time advance $U(x,t)$ on the N equally spaced grid points in the spatial domain $0 < x < L$.
(2) (15 pts) What is the maximum time step that can be used in (1) to maintain numerical stability.
(3) (10 pts) What boundary conditions must be supplied for U.
(4) (10 pts) Can you suggest a modification to the finite-difference scheme to allow stable numerical time integration with a time step larger than what was found in (2)?

OR

2. [45 points]

Evaluate to leading order the integral I(z) for $z \to +\infty$

$$I(z) = \int_0^1 e^{-zt^2} \cos(zt + zt^3) dt.$$