DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 18, 1998

9 a.m. - 1 p.m.

- Answer all problems. Problem 3 has a choice of A or B (answer one only).

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____" and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.
Part I, Question 1
Experimental Quickie
(10 points)

Describe how a Rogowski coil works.
Part I, Question 2
Applied Math
(30 points)

Generals

1. Evaluate the leading asymptotic behavior of \( I(z) \) for \( z \to \infty \) and for \( z \to -\infty \).

\[
I(z) = \int_0^1 e^{z(t^2 - 4t^3 + 3/4)} dt
\]
The ion susceptibility in a non-drifting finite-ion-Larmor-radius Maxwellian magnetized plasma can be written in the following form:

\[
\chi_{xx}^{i} = \sum_{n=1}^{\infty} e^{-\lambda} \frac{n^2 I_n}{\lambda} \frac{2 \omega_p^2}{n^2 \Omega_i^2 - \omega^2},
\]

where \( \lambda = \frac{k_i^2 \kappa T_i}{m \Omega_i^2} \).

(a) (5 points) Write down the kinetic dispersion relation of the **electrostatic** waves for a hot electron, non-drifting Maxwellian magnetized plasma including finite Larmor radius effects only in the \( \chi_{xx}^{i} \) term.

(b) (20 points) Expanding in small \( \lambda \equiv k_i^2 \kappa T_i / m \Omega_i^2 \) and keeping the leading order terms in \( \lambda \), show that the real part of the dispersion relation for the electrostatic waves in the ion cyclotron range of frequency can be reduced to the following quadratic equation to a good approximation,

\[
a k_{\perp}^4 + b k_{\perp}^2 + c = 0
\]

where

\[
a = 3 \omega_p^2 (\kappa T_i / M_i) (4 \Omega_i^2 - \omega^2)^{-1} (\omega^2 - \Omega_i^2)^{-1}
\]

\[
b = + \omega_p^2 / \Omega_i^2 - \omega_p^2 / (\omega^2 - \Omega_i^2) = - \omega_p^2 / (\omega^2 - \Omega_i^2)
\]

\[
c = k^2 (1 - \omega_p^2 / \omega^2 + \omega_p^2 m_e / k^2 \kappa T_e) = \omega_p^2 m_e / \kappa T_e
\]

Hint: \( I_1 \equiv \frac{\lambda}{2} + \vartheta(\lambda^2) \) and \( I_2 \equiv \frac{\lambda^2}{8} + \vartheta(\lambda^4) \).

(c) (10 points) Assuming that \( \Omega_i < \omega < 2 \Omega_i \), give the dispersion relation of \( k_{\perp}(\omega) \) propagating modes (non-FLR and FLR) for \( b^2 >> 4 ac \). Give the name of each mode.

(d) (7 points) Sketch the dispersion relation \( \omega(k_{\perp}) \) of the modes. Going back to the quadratic equation, give the condition for the mode-conversion.

(e) (8 points) Comment on the **direction** of the perpendicular wave phase velocity to that of the group velocity for each branch. What happens to the wave phase velocity and group velocity near the mode-conversion region?
Part I, Question 3b: 
Irreversible Processes [50 points]

Note: Do either question (3a) or (3b)!

In this problem, we will consider a one-component plasma—say, a pure-electron plasma. Thus, don’t worry about interactions between multiple species.

(a) [7 points] The conventional derivation of the plasma collision operator (either the Balescu–Lenard or Landau form) is done in the limit $B \to 0$ ($\omega_c/\omega_p \ll 1$). For the other limit $\omega_c/\omega_p \gtrsim 1$ (equivalently, $\rho/\lambda_D \lesssim 1$), it has been suggested that the proper operator is obtained simply by replacing the Spitzer logarithm with

\begin{equation}
\ln \Lambda \to \ln \left( \frac{\rho}{b_0} \right),
\end{equation}

where $\rho \overset{\text{def}}{=} v_t/\omega_c$ is the gyroradius and $b_0$ is proportional to the distance of closest approach. Give a simple physical argument why that should be so.

(b) [8 points] Now consider that the magnetic field is very strong such that $\rho/\lambda_D \ll 1$. In this limit the above modification (involving perpendicular dynamics only) fails, as it ignores the parallel dynamics of guiding centers. Instead, it is suggested that the perpendicular thermal diffusivity $\kappa$ can be estimated by

\begin{equation}
\kappa \sim \lambda_D^2 \nu_0,
\end{equation}

where $\nu_0$ is the collision frequency for large-angle scattering—i.e., it does not contain $\ln \Lambda$. (Note that this result is independent of $B$.) Give a very short and simple physical argument that supports the estimate (2).

**Hints:** Assume that Eq. (2) is correct. Don’t try to do a serious dynamical calculation. Just argue physically about parcels of heat being randomly exchanged between particles moving along different parallel field lines.

*(Problem continues on next page.)*
(c) [10 points] With the modification (1), the classical scaling for $\kappa$ would be

$$\kappa \sim \rho^2 \nu_0 \ln(\rho/b_0),$$

valid for modest magnetic-field strength. Given that result and formula (2) (valid for very strong $B$), sketch on the following graph the expected behavior of $\kappa$ as a function of

$$\epsilon \overset{\text{def}}{=} (\rho/\lambda_D)^2.$$ 

Calculate and indicate on the graph the value of $\epsilon$ below which the guiding-center result (2) dominates over the classical result (3). (The answer is not exactly $\epsilon = 1$. Be more precise by approximately solving the appropriate transcendental equation in $\epsilon$; briefly justify any nontrivial mathematical steps.)

FIG. 1. Dependence of the cross-field thermal conductivity coefficient $\kappa$ on normalized magnetic field strength $\epsilon \overset{\text{def}}{=} (\rho/\lambda_D)^2 \propto 1/B^2$.

(Problem continues on next page.)
(d) [25 points] In this part, you will be asked to do a calculation using the Test Particle Superposition Principle for parallel-moving guiding centers in a slightly inhomogeneous plasma. The first part of the discussion attempts to place the question into context, but the details may not be essential. Therefore, skim rapidly to find the straightforward question at the end.

Formally, the guiding-center result $\kappa \sim \nu_0 \lambda_D^2$ follows from an appropriate kinetic theory. One might be tempted to obtain that result from the essentially 1D Balescu–Lenard equation

$$\left( \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial v_z} \right) f(x, v_z, t) = \pi \left( \frac{q^2}{m} \right) (\bar{n}q^2) \frac{\partial}{\partial v_z} \int d\bar{v}_z \int \frac{dk_z}{(2\pi)^2} \left[ \frac{(4\pi k_z/k^2)^2}{|D(k, \omega)|^2_{\omega=k_z v_z}} \delta(k_z(v_z - \bar{v}_z)) \right]$$

$$\times \left[ \frac{1}{m^s} \frac{\partial f_s(x, v_z, t)}{\partial v_z} f_s(x, \bar{v}_z, t) - \frac{1}{m^s} \frac{\partial f_s(x, v_z, t)}{\partial \bar{v}_z} f_s(x, v_z, t) \right], \quad (5)$$

where for a pure-electron plasma $s = \tilde{s} = e$. However, one can easily see that the perpendicular heat flow evaluated for this equation vanishes if $f$ is just taken to be a local Maxwellian with $x$-dependent temperature $T(x, t)$ (and no mean flow). One cannot derive Eq. (2) from a conventional Chapman–Enskog approach using the operator (5).

The reason that the heat transport vanishes with the approximate operator (5) is that the usual derivation of the collision operator assumes a coarse-graining in $x$ such that spatial steps in macroscopic transport processes are taken to be large with respect to $\lambda_D$; however, in the present mechanism parcels of heat are exchanged between field lines separated by distances less than $\lambda_D$. The proper generalization of Eq. (5) replaces the standard dielectric shielding $1/|D|^2$ with the following underlined expression, according to

$$\left( \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial v_z} \right) f(x, v_z, t) = \pi \left( \frac{q^2}{m} \right) (\bar{n}q^2) \frac{\partial}{\partial v_z} \int d\bar{v}_z \int \frac{dk_z}{(2\pi)^2} \left[ \frac{(4\pi k_z/k^2)^2}{|D(k, \omega)|^2_{\omega=k_z v_z}} \delta(k_z(v_z - \bar{v}_z)) \right]$$

$$\times \left[ \frac{1}{m^s} \frac{\partial f_s(x, v_z, t)}{\partial v_z} f_s(x, \bar{v}_z, t) - \frac{1}{m^s} \frac{\partial f_s(x, v_z, t)}{\partial \bar{v}_z} f_s(x, v_z, t) \right], \quad (6)$$

where $k_\perp \equiv \{k_y, k_z\}$ (perpendicular to $\tilde{e}$, not $B$) and $\psi$ is the solution to

$$\left[ \frac{\partial^2}{\partial x^2} - k_\perp^2 - \omega_P^2 ik_z \int d\bar{v}_z \frac{\partial f(x, \bar{v}_z)}{\partial \bar{v}_z} \right] \psi(x, k_\perp, \omega; x') = \delta(x - x'). \quad (7)$$

In this problem, you are asked to show some insight into the origins of the underlined expression in Eq. (6). To do so, note the following:

1. Equation (6) is the statistical average of the guiding-center Klimontovich equation

$$\frac{\partial \tilde{N}}{\partial t} + v_z \frac{\partial \tilde{N}}{\partial z} + \frac{q}{m} E_z \frac{\partial \tilde{N}}{\partial v_z} = 0, \quad (8)$$

which defines the guiding-center dynamics (only parallel motion) under consideration.
2. If one focusses on just the velocity-space diffusion term in Eq. (6), at the heart of the operator should be the fluctuation spectrum of the parallel electric field:

\[ \mathcal{E}_{k_1,\omega}(x) \overset{\text{def}}{=} \left\langle |\delta E_z(x, k_{\perp}, \omega)|^2 \right\rangle_{\omega = k_z v_z}. \]  

(9)

Use the Test-Particle Superposition Principle in this slightly inhomogeneous plasma \([f = f(x, v_z)]\) with parallel-moving guiding centers to show that

\[ \mathcal{E}_{k_1,\omega}(x) \propto \int d\bar{x} |\psi(x, k_{\perp}, \omega; \bar{x})|^2 f(\bar{x}, \bar{v}_z). \]  

(10)

Hints:

- You’re not being asked to make a complete calculation of \(\mathcal{E}\); you’re just being asked to show where the explicit terms in Eq. (10) come from. Therefore, don’t worry about normalizations and numerical coefficients.

- However, be sure you explain the conceptual foundations clearly. For example, distinguish between test-particle, induced, and total responses, and discuss why the test particles have the statistical properties you assume.

- If you run out of time, say how you would have proceeded.
Part I, Question 5
Fusion Plasmas
(45 points)

1a. (10 points)
The major toroidal confinement systems are the field-reversed configuration (FRC), the reversed-field pinch (RFP), the spherical torus (ST), the stellarator, and the tokamak. Based on existing devices, rank these configurations in terms of $I_p/B_T$, $\beta_T$, and $\tau_E$. ($I_p$ is the plasma current, $B_T$ the toroidal field, $\beta_T$ the total beta, and $\tau_E$ the energy confinement time). Be quantitative if you can.

1b. (10 points)
What are the advantages and disadvantages (compared to a tokamak) of the FRC, RFP, ST, and stellarator for development as a fusion reactor?

1c. (25 points)
For a tokamak plasma, devise diagnostics to measure three of the following, $n_e$, $T_e$, $T_i$, neutrons, fluctuations, equilibrium, and impurity. Describe:

- Principles of each diagnostic.
- Parameter range over which the diagnostic may be applied.
- Temporal and spatial resolution, e.g. local, chordal, or global. What constrains the resolution?
DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 19, 1998

9 a.m. - 1 p.m.

- Answer all problems.

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Part II, Question 6
Wave Quickie
(10 points)

Consider a plane-polarized wave propagating along the magnetic field in a positronium plasma (equal numbers of positrons and electrons). Does it undergo Faraday rotation? Explain briefly.
Computational Methods
Consider the following computational problem

\[ A \cdot x = y \]  \hspace{1cm} (1)

where \( A \) is an \( n \times n \) matrix, and \( x \) and \( y \) are \( n \)-long column vectors; for given \( A \) and \( y \), we want to determine \( x \). For the case when \( A \) has the special form

\[
A = \begin{pmatrix}
a_{11} & a_{12} & 0 & 0 & \ldots & 0 \\
0 & a_{22} & a_{23} & 0 & \ldots & 0 \\
0 & 0 & a_{33} & a_{34} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & a_{nn}
\end{pmatrix} \hspace{1cm} (2)
\]

(i.e., \( a_{ij} = 0 \) unless \( i = j \) or \( i + 1 = j \)), outline a numerical algorithm for finding \( x \). Given an estimate of the number of arithmetical operations needed.
Magnetohydrodynamics

Prove that there is no static axisymmetric equilibria with only toroidal magnetic field that is "confined" by an axisymmetric boundary with circular cross section. That is: in cylindrical coordinates $R, \Theta, Z$ show that

$$\mathbf{j} \times \mathbf{B} = \nabla p \tag{1}$$

has no solution with

$$\mathbf{B} = B_T(R, Z)\hat{\Theta} \tag{2}$$

where $\hat{\Theta}$ is a unit vector in the $\theta$ direction and

$$p = p(R, Z) \tag{3}$$

where $p$ vanishes on the boundary

$$(R - R_0)^2 + Z^2 = a^2. \tag{4}$$

Hint: Write out the cylindrical components of the magnetostatic equations and show that these imply that $RB_T = F(p)$. Then show that $4\pi R^2 = -F'(p)F(p)$, and draw the proper conclusions.
Basic plasma physics.

In this problem you will derive the dispersion relation for small amplitude electrostatic plasma waves in an unmagnetized plasma, using fluid equations for the electrons and assuming stationary ions.

(a) 5 points. Write down the fluid equations for one dimensional dynamics of the electrons. Close the fluid equations by assuming a simple equation of state for the electron pressure:

$$p = p_0 (n/n_0)^\Gamma$$

where $p_0$ and $n_0$ are the equilibrium electron pressure and density, and $p = p_0 + p_1$ and $n = n_0 + n_1$ are the total electron pressure and density.

(b) 20 points. Linearize these fluid equations and then derive a dispersion relation for electron plasma waves. Sketch a plot of the frequency $\omega$ vs. the wave number $k$.

(c) 10 points. Suppose some disturbance occurs in the vicinity of $x = 0$ which causes electron plasma waves. How long will it take before an antenna at $x = L$ will detect these waves? What is this delay time in the limit that the electron temperature goes to zero?

(d) 10 points. Explain why Landau damping for these waves is small if $k\lambda_{De} \ll 1$, where $k$ is the wave number of the wave and $\lambda_{De}$ is the electron Debye length.
Part II, Question 10
Transport [25 points]

Consider classical transport theory for strongly magnetized, quasi-neutral plasma. Begin with the simplified, steady-state momentum equation

\[ 0 = (nq)_s (E + c^{-1} u_s \times B) - \nabla P_s - \sum_{\bar{s}} (mn)_{s,\bar{s}} \nu_{s,\bar{s}} (u_s - u_{\bar{s}}) \]  

(1)

and find a general expression (valid for an arbitrary number of ion species) for the cross-field collisional flux

\[ \Gamma_s \overset{\text{def}}{=} n_s u_{\perp,s} \]  

(2)

Then prove that for a quasi-neutral plasma with just one (fully stripped) ion species, say helium, one has intrinsic ambipolarity—that is,

\[ \Gamma_{\perp,e} = Z \Gamma_{\perp,i} \]  

(3)

where \( Z \) is the atomic number. Assume a spatially constant and species-independent temperature.

**Hints:** Appropriately iterate Eq. (1) in the limit of strong magnetic field, and use conservation of momentum density during collisions.
Experimental methods

Starting at $t = 0$, moderate-strength RF power at frequency $\omega$ is applied to helium gas in a long, thin (length $>>$ radius) cylindrical glass tube. The RF power is coupled via an antenna which is a wire helically wound N-turns around the cylinder. The antenna has a length $L$ and radius $r$. A partially ionized unmagnetized helium plasma is formed and steady-state achieved at time $t = t_0$. The helium gas has a temperature-dependent ionization rate coefficient $K_{ii}(T_e)$. The density, $n_e$, is such that $\omega_{pe} > \omega$ in steady state.

a) In the initial state (no plasma) calculate the average axial and azimuthal electric field strengths inside the glass tube produced by RF currents in the antenna.

b) The electric fields cause the plasma to form. When the density becomes so large that $\omega < \omega_{pe}$ the plasma interior is shielded from the electric fields. What is the penetration distance for each component?

C) Describe the important processes in the steady state. Assume that the plasma flows to the walls at the ion acoustic speed. Show how the electron temperature depends on the neutral gas density and the radius of the tube.

d) Describe a diagnostic you would use to measure the density.