DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 10, 1999

9 a.m. - 1 p.m.

- Answer all problems. Problem 4 has a choice of A or B (answer one only).

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Problems for Part I, May 10, 1999

1)	Whistler Waves	30 points
2)	Irreversible Processes	50 points
3)	Experimental	20 points
4)	Mathematical Method	35 points
	Do problem 4A **OR** 4B	
5)	Drift Waves	45 points
		Total - 180 points

Monday, May 10

Part I

Question I

Whistler Waves [30 points total]

During World War I, the Germans detected ionospheric whistler waves while attempting to intercept Allied radio transmissions. The whistlers were heard as a series of declining tones. Later studies demonstrated that whistlers are excited by lightening and can propagate from between the northern and southern hemispheres. Given that the dispersion relation for whistler waves can be approximated as:

$$n^2 = \frac{\omega_{pe}^2}{|\omega\Omega_{ce}\cos\theta|}$$

when $|\Omega_{ce}\cos\theta| >> \omega$ and when $n^2 >> 1$, explain :

(a) [10 points] the unique sound characteristic of the whistler waves (assume $\theta \cong 0°$);

(b) [20 points] why the whistlers propagate between the northern and southern hemispheres.

Monday, May 10, 1999
Part I
Question 2

Irreversible processes [50 minutes]

Consider a population of *test* electrons in an magnetized electron-ion plasma. Assume that all distributions are symmetric about the direction of the magnetic field. Provided that the velocity of the test electrons is sufficiently large, the distribution, f, of the test electrons evolves according to

$$\frac{\partial f}{\partial t} - \Gamma\Big(\frac{1}{v^2}\frac{\partial}{\partial v}f + \frac{1+Z}{2v^3}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu}f\Big) = 0, \qquad (1)$$

where Z is the ion charge state, $\Gamma = ne^4 \ln\Lambda/4\pi\epsilon_0^2 m_e^2$, $\mu = \cos\theta$, and v and θ are the magnitude and direction of the electron velocity $\mathbf{v}$.

(a) [15 minutes] Describe physically the terms in equation (1). Discuss the approximations and assumptions made in its derivation. *Sketch* the evolution for f from an initial delta function.

(b) [15 minutes] An alternative description of the test electrons is with the Langevin equations for a particular test electron. Show that the Langevin equations corresponding to equation (1) are

$$\frac{dv}{dt} = -\frac{\Gamma}{v^2}, \qquad \frac{d\mu}{dt} = A(t), \qquad (2)$$

where $A(t)$ is a stochastic term satisfying

$$\langle A(t)\rangle = -\Gamma\frac{1+Z}{v^3}\langle\mu\rangle, \qquad \langle A(t)A(t')\rangle = \Gamma\frac{1+Z}{v^3}\left\langle 1-\mu^2\right\rangle\delta(t-t'). \qquad (3)$$

(c) [10 minutes] Assume that at $t=0$, the test electrons satisfy $v(t=0)=v_0$ and $\mu(t=0)=\mu_0$. Show that the mean parallel velocity, $\langle v\mu\rangle$, satisfies

$$\frac{d\langle v\mu\rangle}{dt} = -\Gamma\frac{2+Z}{v^3}\langle v\mu\rangle. \qquad (4)$$

[Hint: note that v is a non-stochastic variable.]

(d) [10 minutes] Find an explicit expression of $\langle v\mu\rangle$. [Hint: solve for v and substitute into equation (4).]

Monday, May 10, 1999
Part I
Question 3

Describe a diagnostic for measuring the ion temperature of a hot magnetically confined plasma. Describe a complication to the interpretation of the measurement.

Monday, May 10, 1999
Part I
Question 4A
(choose 4A OR 4B)

Finite Difference Equations (35 min)

Consider the system of equations for $v(t,x)$ and $p(t,x)$ with c being a constant:

$$\frac{\partial v}{\partial t} = \frac{\partial p}{\partial x}, \tag{1a}$$

$$\frac{\partial p}{\partial t} = c^2 \frac{\partial v}{\partial x} \tag{1b}$$

and consider the corresponding finite difference scheme

$$v_j^{n+1} = v_j^n + \frac{\delta t}{2\delta x}\left[p_{j+1}^n - p_{j-1}^n\right] \tag{2a}$$

$$p_j^{n+1} = p_j^n + \frac{c^2 \delta t}{2\delta x}\left[v_{j+1}^n - v_{j-1}^n\right] \tag{2b}$$

where $v_j^n \equiv v(n\delta t, j\delta x)$, $p_j^n \equiv p(n\delta t, j\delta x)$, with δt and δx being the (constant) time step and space step, respectively.

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(1) (20min) Show that the finite difference method (2a),(2b) leads to a numerical instability for finite values of $\delta t/\delta x$.

(2) (15 min) Suggest a modification to the finite difference scheme (2a),(2b) that is stable and give the conditions on δt and δx for stability.

Monday, May 10, 1999
Part I
Question 4B
(choose either 4A OR 4B)

35 points

1. Consider

$$x\frac{d^2y}{dx^2} + (a - x)\frac{dy}{dx} + by = 0$$

25 points (a) Find integral representations for two linearly independent solutions.

10 points (b) What are conditions on a and b such that these representations exist?

1. Drift waves, basic plasma physics (45 points).

(a) (25 points) In this problem you will derive the basic dispersion relation for electrostatic drift waves in a slab plasma with a uniform magnetic field $\vec{B} = B\hat{z}$, and an equilibrium distribution function F_0 that is Maxwellian with a density gradient such that $\nabla F_0 = -\hat{x}F_0/L_n$. Starting with a simple drift-kinetic equation for the ions (keeping just the $E \times B$ drift and parallel electric field, and ignoring the ion polarization drift or ion polarization density), show how to derive the perturbed ion density

$$\tilde{n_i} = -n_{i0}\frac{e\phi}{T_e}\left[\frac{T_i}{T_e}\left(1+\zeta Z(\zeta)\right)+\frac{\omega_*}{|k_z|\sqrt{2}v_{ti}}Z(\zeta)\right]$$

where $\zeta = \omega/(|k_z|\sqrt{2}v_{ti})$, $v_{ti} = \sqrt{T_i/m_i}$, and Z is the plasma dispersion function as defined in the NRL formulary. [You can ignore the subtleties of the analytic continuation of the Z function, and just assume in your derivation that $k_z > 0$ and $\mathrm{Im}(\omega) > 0$ so that the velocity integrals can be easily related to the standard form of the Z function.]

(b) (20 points) Expand the above expression for $\tilde{n_i}$ for $\zeta \gg 1$, to derive a quadratic equation in ω. In deriving this dispersion relation, assume quasineutrality $\tilde{n_i} = \tilde{n_e}$ (for simplicity ignoring any ion polarization density correction to this quasineutrality equation), and using an adiabatic electron response $\tilde{n_e} \propto \phi$.

Solve the quadratic equation for ω, and plot the two roots vs. $k_z c_s$ (where $c_s = \sqrt{T_e/m_i}$), illustrating the regime where the drift wave exists and the regime where sound waves dominate. Briefly state when ion Landau damping would become important.

DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 11, 1999

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Problems for Part II, May 11, 1999

1)	Collisionless Damping	20 points
2)	Experimental	20 points
3)	Neoclassical theory	50 points
4)	MHD	45 points
5)	Basic Plasma	45 points
		Total - 180 points

Tuesday, May 11
Part II
Question 1

Collisionless Damping [20 points]

Explain what is meant by collisionless damping and indicate the parameter range in which both nonlinear effects and collisional effects may be ignored.

Tuesday, May 11, 1999

Part II

Question 2

(20 points)

Explain the differences between a single and a double Langmuir probe. What plasma conditions might warrant the use of one over the other. In your discussion, note when the I-V characteristics are similar.

Tuesday, May 11, 1999
Part II
Question 3

NEOCLASSICAL THEORY: (50 points)

This problem deals with the Coulomb collisional relaxation of a tokamak plasma driven by a pressure gradient in the radial direction and by a parallel electric field, $E_{\|}$. Consider the associated "neoclassical" transport in the long mean-free-path "banana regime." For simplicity, ignore temperature gradients.

(A) 30 points

What is the particle flux in terms of these driving forces? Specifically, provide a simplified (heuristic) derivation of the coefficients for the pressure gradient $(-T_e dn_0/dr)$ and $-n_0 E_{\|}$ driving forces.

(B) 10 points

How is the classical (Spitzer) conductivity, $\sigma_{\|}$, modified in this regime?

Give a simplified (heuristic) derivation for the neoclassical current density – again specifying the coefficients for the driving forces.

(C) 10 points

What is Onsager Symmetry as applied to this problem? How can it be usefully applied in writing down the solutions in parts (A) and (B)?

Tuesday, May 11, 1999

Part II

Question 4

MHD Problem(45 points)

Consider MHD waves propagating in the x direction perpendicular to a uniform B_0 field in the z direction. $\mathbf{k}$ is in the x direction. The independent variables are: $\rho_1, p_1, B_{1y}, B_{1z}, v_{1x}, v_{1y}, v_{1z}$, so that there must be seven independent waves. Two are magnetosonic waves while four are degenerate limits of the slow and intermediate waves (i.e. $\omega = 0$) and the seventh is an entropy wave. (All equilibrium qunatities are uniform.)

You may take as given that the perturbation of the ideal equations leads to the equations

$$\omega\rho_1 = \rho_0 k v_{1x} \tag{1}$$

$$\omega p_1 = \gamma p_0 k v_{1x} \tag{2}$$

$$\omega B_{1z} = B_0 k v_{1x} \tag{3}$$

$$\omega\rho_0 v_{1x} = k p_1 + k B_0 B_{1z}/4\pi \tag{4}$$

$$\omega v_{1y} = \omega v_{1z} = \omega B_{1y} = 0. \tag{5}$$

where perturbed quantities are proportional to $e^{-i\omega t + ikx}$.

1. (15 points)
Find the two normal modes for the magnetosonic modes (i.e. $\omega = \pm\omega_s$) propagating in the $\pm x$ direction. Find five other normal modes giving values of the seven independent variables for each of these modes. (You may express the results for the ms mode in terms of ω_s.)

2. (15 points)
Start with an initial perturbation at $t = 0$ of $B_{1z} = B_{init}, \rho_1 = p_1 = B_{1y} = v_{1x} = v_{1y} = v_{1z} = 0$. Show that two magnetosonic waves and two other zero frequency wave modes are developed. (Hint: expand the initial perturbation in the normal modes.)

3. (15 Points)
Find the velocity amplitude of the resulting magnetosonic mode propagating in the $+x$ direction.

Tuesday, May 11

Part II, Question 5

Basic Plasma (45 points)

Consider a plasma immersed in a z-directed magnetic field. Assume that there is a source of electrons, so that under certain circumstances the electron velocity distribution function, f_e, might be described rather well by the following equation:

$$\frac{\partial f}{\partial t} = \Gamma \left[\frac{1}{v^2} \frac{\partial}{\partial v} f + \frac{1 + Z_i}{2v^3} \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial}{\partial \mu} f \right] + S(v, \mu, t), \qquad (*)$$

where Γ is a constant, v is the electron speed, $\mu = v_z/v$ is the velocity pitch angle, Z_i is the ion charge state, and S is the source function.

Suppose also that this plasma is fueled somehow so that electrons are produced with a tendency to have more energy perpendicular to the magnetic field than parallel to it. The fueling might be described by a source term

$$S = (1 - \mu^2)\dot{g}(v).$$

Note that $\dot{g}(v)$ depends only on electron speed.

(a) (5 pts) What are the conditions under which equation (*) is a good description of electron collisions in an infinite homogeneous plasma?

(b) (10 pts) Does the collision term in equation (*) conserve particles, entropy, or energy? Explain.

(c) (10 pts) What is the steady state distribution of electron speeds?.

(d) (15 pts) What is the deviation from isotropy in velocity space as $t \to \infty$? At what speeds would the deviation from isotropy tend to be most pronounced?

(e) (5 pts) What is the steady state electron velocity distibution for $\dot{g}(v) = k\delta(v - v_0)$, where k is a constant?

Legendre harmonics: $P_0 = 1$, $P_1 = \mu$, $P_2 = (3\mu^2 - 1)/2$.

Helpful relation: $1 - \mu^2 = 2(P_0 - P_2)/3$.