DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 15, 2000

9 a.m. - 1 p.m.

- Answer all problems. Problem 7 has a choice of A or B (answer one only).

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate <u>specifically</u> how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Problems for Part I, May 15, 2000

1)	Tokamak orbits	20 points
2)	Mirror orbits	20 points
3)	Experimental quickie	15 points
4)	Flux function quickie	10 points
5)	Rayleigh-Taylor instability	50 points
6)	Transport quickie	10 minutes
	Do problem 7A **OR** 7B	
7)	a) Transport theory	40 minutes
	b) Z >>1 Ion acoustic Wave Landau damping	
8)	Tunnelling/Stokes applied math	15 points

Total - 180 points

Part I, Question 1

Tokamak Orbits [20 points]

(a) (5 pts)

What are the constants of the motion for particle orbits in a tokamak?

(a) (10 pts)

Sketch the orbits of co-passing, counter-passing, and banana trapped ions in a tokamak.

(c) (5 pts)

Write an approximate expression for the banana width of a trapped particle in terms of its initial parallel velocity and the poloidal magnetic field. Explain brifly how you arrive at this expression.

Part I, Question 2

Mirror Orbits [20 points]

(a) (5 pts)

Describe briefly how mirror confinement works, and sketch single particle orbits. Be sure to note all drift motions.

(b) (10 pts)

Derive a trapping condition for mirror confined particles in terms of the particle's initial coordinates and the mirror ratio.

(c) (5 pts)

Please explain briefly what fluid instability might be exhibited by a plasma confined in a simple mirror?

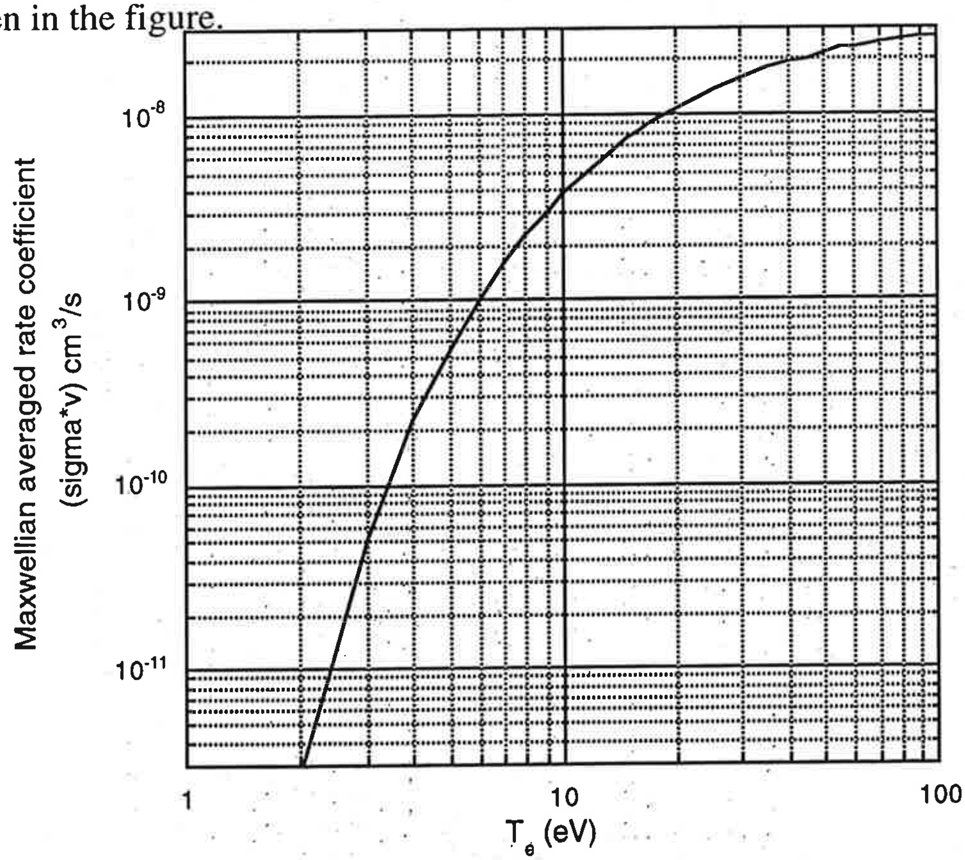
Experimental quickie (15)

Consider a low-temperature (T_e ~ 1 - 100 eV, T_i ~ 0 eV) isothermal argon plasma between two electrodes, a cathode and an anode. Other than the electrodes, the plasma touches no boundaries in the system. A volumetric auxiliary heating method, independent of the weak DC electrode bias voltage, ~ 5 T_e, sustains the plasma by heating the electrons. (The heating method could be microwave absorption.)

The system has the following parameters:

Cathode area, A = 2 cm^2,
Plasma volume, V = 30 cm^3,
Neutral argon pressure = 0.3 mTorr (n_0 ~10^{13} cm^{-3}), and
Cathode current, I = 0.1 ampere.

Estimate the ion density and electron temperature (to ~20% accuracy) using an iterative procedure. The ionization rate coefficient for neutral argon is given in the figure.



Part I, Question 4

Flux Function (10 points)

Representation of the magnetic field in terms of a flux function. Consider a magnetic field in slab geometry with $\mathbf{B} = \mathbf{B}(x, y)$ independent of z. Starting from the vector potential, $\nabla \times \mathbf{A}(x, y) = \mathbf{B}$, derive an expression for $\mathbf{B}$ in terms of a flux function ψ. Show that the magnetic field lines lie in surfaces of constant ψ.

Part I, Question 5
Rayleigh-Taylor Instability (50 points)

Consider a plasma in slab geometry supported in equilibrium against a gravitational force -$\rho g \hat{x}$ by a magnetic field $\mathbf{B} = B(x)\hat{z}$.

(a) (5 pts) Calculate the one-dimensional radial force balance relation describing the equilibrium. Assume $\nabla p = 0$ in equilibrium.

In the remainder of this problem you will be asked to proceed step-by-step through a derivation of the eigenvalue equation describing the linear stability of the equilibrium. Suggestions for simplifications along the way will help you get through the calculation in the allotted time. We will assume that the perturbations vary as $\xi = \xi(x)\exp(-i\omega t + ik_y y)$. ($k_z = 0$). To simplify the analysis, we will use $\nabla \cdot \mathbf{v} = 0$ for the equation of state.

(b) (8 pts) Give expressions for the components of the perturbed field, B_{1x}, B_{1y}, B_{1z}, in terms of the displacement, ξ, and equilibrium quantities. Use $\nabla \cdot \mathbf{v} = 0$ to simplify the expression.

(c) (5 pts) Express ρ_1, the perturbed density, in terms of ξ and equilibrium quantities, again using $\nabla \cdot \mathbf{v} = 0$.

(d) (12 pts) Give the x component of the momentum equation, and simplify it by using the equilibrium equation.

(e) (5 pts) Give the y and z components of the momentum equation.

(f) (7 pts) Eliminate all perturbed quantities other than ξ_x to obtain a single one-dimensional eigenvalue equation in terms of ξ_x and equilibrium quantities.

(g) (8 pts) Now simplify the eigenvalue equation to the case where the eigenfunction varies slowly, $\partial\xi_x/\partial x \ll k_y\xi_x$, $\partial^2\xi_x/\partial x^2 \ll k_y^2\xi_x$, and where the equilibrium density profile $\rho_0(x) = c_0 e^{(x/L)}$, with $1/L \ll k_y$. Obtain an expression for the growth rate. What happens to the growth rate if L is negative?

Part I, Question 6:
Transport [10 points]

For transport in a neutral electron–ion plasma across a strong magnetic field $\boldsymbol{B}$, the particle diffusion is intrinsically ambipolar, $D_e = D_i$, and does not depend on the ion mass; one can take $m_i \to \infty$. For that same limit of infinite ion mass, suppose that $\boldsymbol{B} = 0$. What happens to the particle diffusion of the plasma fluid? Does it exist? If it does, is it ambipolar? If it does not, what physical process(es) replace it?

Part I, Question 7A:
Kinetic theory and transport [~~35~~ 40 points]

According to Braginskii, the velocity-driven friction force $\boldsymbol{R}$ in the presence of a strong magnetic field is anisotropic: for $Z = 1$, one has $R_\parallel \propto 0.51$, $R_\perp \propto 1$. Thus the perpendicular resistivity $\eta_\perp$ is greater than the parallel resistivity $\eta_\parallel$.

(a) [**5 points**] Give a simple physical argument why $\eta_\perp > \eta_\parallel$.

In the remainder of the problem you will study one method for actually determining $\eta_\perp$ from kinetic theory, for an electron–ion, $Z = 1$ plasma.

- For $\boldsymbol{B} = 0$, the most direct way of determining resistivity would be to impose a current $\boldsymbol{j}$, determine the resulting $\boldsymbol{E}$, then take the ratio of $\boldsymbol{E}$ and $\boldsymbol{j}$ to get η from Ohm's law $\boldsymbol{E} = \eta \boldsymbol{j}$.
- But for $\boldsymbol{B} \neq 0$, it's actually easier to do it the other way around: start with $\boldsymbol{E}$, determine $\boldsymbol{j}$, and extract the appropriate coefficient in the tensor relation between $\boldsymbol{j}$ and $\boldsymbol{E}$. A problem with this approach is that if both species are free to move, they will both move with the $\boldsymbol{E} \times \boldsymbol{B}$ velocity and the friction force will vanish to lowest order, making it difficult to find the effect due to resistivity.
- A trick to be used in the remainder of the problem is to decree that *the ions do not move at all.* Then the friction force does not vanish and one can easily find the desired resistivity coefficient.

(b) [**10 points**] Impress a d.c., infinite-wavelength perpendicular electric field $\boldsymbol{E}_\perp$ on a strongly magnetized plasma ($\nu/\omega_c \ll 1$) *in which the ion fluid velocity is somehow constrained to vanish:* $\boldsymbol{u}_i = 0$. Assume that the frictional drag in the electron momentum equation has the form $-\nu \boldsymbol{u}_\perp$ (where $\boldsymbol{u} = \boldsymbol{u}_e$, since $\boldsymbol{u}_i = 0$). Using the simplest possible form of the momentum equation, show as succinctly as possible that the resulting perpendicular current has the form

$$\boldsymbol{j}_\perp \approx \left(\frac{\omega_p^2}{4\pi\nu}\right)\Bigg[\underbrace{\left(\frac{\nu}{\omega_c}\right)\boldsymbol{E}_\perp \times \widehat{\boldsymbol{b}}}_{\boldsymbol{j}_\perp^{(1)}} + \underbrace{\left(\frac{\nu}{\omega_c}\right)^2 \boldsymbol{E}_\perp}_{\boldsymbol{j}_\perp^{(2)}}\Bigg], \tag{1}$$

where all quantities refer to electrons and ω_c is signed.

(c) [~~20~~ 25 **points**] Equation (1) gives the form of the response to $\boldsymbol{E}_\perp$, but the value of ν remains to be determined. [Note that in Eq. (1) the first-order term does not actually involve ν, but the second-order term does, so by comparing the direct solution of the kinetic equation with the form of $\boldsymbol{j}_\perp^{(2)}$, one can determine the proper value of ν.]

By ordering $\omega_c = \mathcal{O}(\epsilon^{-1})$ (where $\epsilon \stackrel{\text{def}}{=} \nu/\omega_c \ll 1$):

- Show how to do perturbation theory on the kinetic equation (assuming a global Maxwellian background) so as to determine $\boldsymbol{j}_\perp^{(1)}$ and $\boldsymbol{j}_\perp^{(2)}$. First set up the zeroth-, first-, and second-order equations for the distribution function.
- Solve the first-order equation and verify that you recover the $\boldsymbol{j}_\perp^{(1)}$ of Eq. (1).
- In the remaining time, discuss how you would solve the second-order equation and find $\boldsymbol{j}_\perp^{(2)}$. Carry through the calculation to the extent you have time, but don't exceed the 25 minutes alloted for this part of the problem. However, do show *mathematically* that electron–electron collisions do not contribute to $\boldsymbol{j}_\perp^{(2)}$. (In that proof, you may use known properties of C_{ee} and C_{ei}.)

Part I, Question 7B
Electron Landau damping of ion acoustic waves [40 points]

[Parts (b-d) can be done independently of part (a).]

(a) (15 pt) Starting from the Vlasov equation in an unmagnetized plasma, show that the dispersion relation for electrostatic waves assuming quasineutrality can be written in the form

$$0 = \sum_s \frac{\omega_{ps}^2}{v_{ts}^2} \left[1 + \zeta_s Z(\zeta_s)\right]$$

where s denotes particle species, $v_{ts}^2 = T_s/m_s$, $\zeta_s = \omega/(|k|v_{ts}\sqrt{2})$, and Z is the plasma dispersion function, and the other notation is standard. [In your derivation, you can ignore the subtleties of the analytic continuation of the Z function. Just show how the velocity integrals can be related to the standard form of the Z function for $Im(\omega) > 0$ and rely on the Z function being defined to properly continue the solution to arbitrary ω.]

(b) (10 pt) Consider the case of two species: ions of charge Z and electrons. Expand this dispersion relation to lowest non-trivial order in the limit $v_{ti} \ll \omega/|k| \ll v_{te}$ to get the lowest order dispersion relation for ion acoustic waves. You can ignore higher order terms involved with electron Landau damping in this part of the problem.

(c) (5 pt) Show that $v_{ti} \ll \omega/|k|$ is a good approximation for $Z \gg 1$ even if $T_i \sim T_e$. Provide a brief physical argument about whether ion Landau damping will be important in this case.

(d) (10 pt) Now repeat step (b), but keep the next order correction needed to get electron Landau damping, in the limit of weak damping. Writing the frequency as $\omega = \omega_0 + \delta\omega$, where ω_0 is the lowest order ion acoustic-wave frequency you found in part(b), calculate the relative damping rate $i\delta\omega/|\omega_0|$.

(This problem is important in certain laser-plasma instabilities.)

Part I, Question 8

Tunnelling/Stokes applied math (15 points)

Consider the wave equation

$$\frac{d^2\psi}{dz^2} + z^2\psi = 0$$

Draw a Stokes diagram and calculate transmission and reflection coefficients for an incident wave from the left. The Stokes constant for a second order zero is $S = \sqrt{2}i$.

DEPARTMENT OF ASTROPHYSICAL SCIENCES
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GENERAL EXAMINATION, PART II

MAY 16, 2000

9 a.m. - 1 p.m.

- Answer all problems. Problem 6 has a choice of A or B (answer one only)..

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

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Problems for Part II, May 16, 2000

1)	Magnetic field diffusion quickie	10 points
2)	Parametric applied math	45 points
3)	Bernstein wave	45 points
4)	Nonlinear dynamics	55 points
5)	Diagnostics	15 points
	Do problem 6A **OR** 6B	
6)	a) Wave quickie	15 points
	b) Tokamak essay	
		Total - 185 points

Part II, Question 1
Magnetic field diffusion quickie [10 points]

Write down an expression for the magnetic field diffusion coefficient. Approximately how far can the magnetic field diffuse in a time $1/\nu_{ei}$ (where ν_{ei} is the electron-ion collision frequency). Express your result relative to the collisionless skin depth. For $\beta \sim 10\%$, is this distance small or large compared to the ion gyroradius?

Part II, Question 2

Parametric applied Math (45 points)

A complex three-wave parametric instability mode has the form

$$\psi(t) = t^2 \int_0^1 dw \left(\frac{w}{1-w}\right)^{i\lambda} e^{-4i\lambda w t^2}$$

with t = time and $\lambda \gg 1$.

Find the time dependence of $\psi(t)$ for early ($t \ll 1$) and late ($t \gg 1$) times. Numerical factors are secondary, first find the time dependence.

Part II, Question 3

Wave-driven density perturbations [45 points total]

a) [15 points] Derive an expression for the perturbed density of species, s, driven by a general electromagnetic wave in a uniformly magnetized, homogeneous plasma in terms of the plasma susceptibility, $\overleftrightarrow{\chi}_s$, for species, s.

b) [15 points] For a quasineutral plasma consisting of a single ion species and electrons, derive an approximate form for the ion Bernstein wave dispersion relation (assuming purely perpendicular propagation, $T_e \sim T_I$, and retaining only the lowest order contributions from electrons in the electrostatic limit).

Hint:

$$\varepsilon_{xx} = 1 - \sum_s \frac{4\pi n_s m_s c^2}{B_0^2} \frac{\alpha(\nu_s, \lambda_s)}{\lambda_s}$$

$$\alpha(\nu_s, \lambda_s) = 2 \sum_{n=1}^{\infty} e^{-\lambda_s} I_n(\lambda_s) \frac{n^2}{\nu_s^2 - n^2}$$

$$\nu_s = \frac{\omega}{\Omega_s}$$

$$e^{-\lambda} I_n(\lambda) = \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n \left[1 - \lambda + \left(\frac{\lambda}{2}\right)^2 \left(2 + \frac{1}{n+1}\right) + \cdots\right] \qquad , \lambda \ll 1$$

$$e^{-\lambda} I_n(\lambda) = \left(\frac{1}{(2\pi\lambda)}\right)^{1/2} \left[1 - \frac{4n^2 - 1}{8\lambda} + \cdots\right] \qquad , \lambda \gg 1$$

c) [15 points] Use the results of parts I and II to demonstrate that the quasineutrality condition is preserved through first order in the electron and ion density perturbations driven by this wave in a typical tokamak plasma. Comment on the relative magnitude of the perturbed ion density to the equilibrium ion density.

Part II, Question 4:
Nonlinear processes and Fokker–Planck equations [55 points]

*Scan the entire problem before you begin. Each of the parts is independent (except for common notation). The principal things you are asked to show are in **boldface italics**.*

The Hasegawa–Mima equation in the absence of a background density gradient ($\omega_* = 0$) is

$$(1-\nabla^2)\frac{\partial \varphi}{\partial t} + \boldsymbol{V}_E \cdot \nabla(-\nabla^2 \varphi) = 0, \qquad \text{where} \qquad \boldsymbol{V}_E \stackrel{\text{def}}{=} \widehat{\boldsymbol{z}} \times \nabla\varphi. \tag{1a,b}$$

Because of the nonlinearity, short-scale fluctuations can beat together to give long-scale ones, and *vice versa*. Thus, one can write

$$\varphi = \underbrace{\overline{\varphi}}_{\substack{\text{long}\\ \text{wavelengths}}} + \underbrace{\varphi^{\text{mid}}}_{\substack{\text{intermediate wavelengths}\\ \text{(ignored in this problem)}}} + \underbrace{\breve{\varphi}}_{\substack{\text{short}\\ \text{wavelengths}}}, \tag{2}$$

where, for example,

$$\overline{\varphi}(\boldsymbol{x},t) = \sum_{\boldsymbol{q}\text{ small}} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\varphi_{\boldsymbol{q}}(t), \qquad \breve{\varphi}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}\text{ large}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\varphi_{\boldsymbol{k}}(t). \tag{3}$$

(It's conventional to use $\boldsymbol{q}$ when referring to the long wavelengths, and to use $\boldsymbol{k}$ when referring to the short wavelengths.) In this problem you will be taken step by step through the use of Fokker–Planck methods and other arguments to discuss how the long scales $\overline{\varphi}$ affect the short scales $\breve{\varphi}$.

(a) [7 points] Suppose that one has somehow initially excited φ fluctuations of short wavelength (large wave number $\boldsymbol{k}$): $\varphi(t=0) = \breve{\varphi}$. It is claimed that the nonlinear Hasegawa–Mima equation can then spontaneously generate long-wavelength fluctuations $\overline{\varphi}$ (wave number $\boldsymbol{q} \ll \boldsymbol{k}$). By Fourier-transforming a general advection term of the form $\boldsymbol{V}(\boldsymbol{x}) \cdot \nabla\psi(\boldsymbol{x})$, where $\boldsymbol{V}$ and ψ are arbitrary functions, ***explain how small $\boldsymbol{q}$'s can be generated by the nonlinearity.***

(b) [5 points] Now suppose that long-wavelength fluctuations $\overline{\varphi}$ have indeed been generated, giving rise to a large-scale velocity field $\overline{\boldsymbol{V}} = \widehat{\boldsymbol{z}} \times \nabla\overline{\varphi}$. The advective effect due to $\overline{\boldsymbol{V}}$ will react back on the short scales $\breve{\varphi}$. From Eq. (1a), the relevant advection equation is

$$(1-\nabla^2)\frac{\partial \breve{\varphi}}{\partial t} + \overline{\boldsymbol{V}} \cdot \nabla(-\nabla^2 \breve{\varphi}) = 0. \tag{4}$$

For constant $\overline{\boldsymbol{V}}$, argue that the small scales evolve according to

$$\frac{\partial \breve{\varphi}_{\boldsymbol{k}}}{\partial t} + i\Omega_{\boldsymbol{k}}\breve{\varphi}_{\boldsymbol{k}} = 0, \qquad \text{where} \qquad \Omega_{\boldsymbol{k}} \stackrel{\text{def}}{=} \left(\frac{k^2}{1+k^2}\right)\boldsymbol{k}\cdot\overline{\boldsymbol{V}}. \tag{5a,b}$$

(c) [5 points] More generally, $\overline{\boldsymbol{V}}$ really varies slightly on long space scales $\boldsymbol{X}$ and slow time scales T: $\overline{\boldsymbol{V}} = \overline{\boldsymbol{V}}(\boldsymbol{X},T)$, where

$$\overline{\boldsymbol{V}}(\boldsymbol{X},T) = \sum_{\boldsymbol{q}\text{ small}} e^{i\boldsymbol{q}\cdot\boldsymbol{X}}\widehat{\boldsymbol{z}} \times [i\boldsymbol{q}\varphi_{\boldsymbol{q}}(T)]. \tag{6}$$

[We follow a standard convention and use $(\boldsymbol{X},T)$ rather than $(\boldsymbol{x},t)$ in order to emphasize the long-wavelength, low-frequency nature of $\overline{\boldsymbol{V}}$.] Thus one has the ray equations

$$\frac{d\boldsymbol{X}}{dT} = \frac{\partial\Omega_{\boldsymbol{k}}(\boldsymbol{X},T)}{\partial\boldsymbol{k}}, \qquad \frac{d\boldsymbol{k}}{dT} = -\frac{\partial\Omega_{\boldsymbol{k}}(\boldsymbol{X},T)}{\partial\boldsymbol{X}}. \tag{7a,b}$$

Explain in simple qualitative terms the physical significance of the ray equations. (This is a general physics question. The answer has nothing to do with the Hasegawa–Mima equation *per se* or with nonlinearity.)

(Problem continues on next page.)

(d) [8 points] ***Show that the short-scale advection equation***

$$(1-\nabla^2)\frac{\partial\breve{\varphi}}{\partial t}+\overline{\boldsymbol{V}}\cdot\nabla(-\nabla^2\breve{\varphi})=0 \qquad (\nabla\cdot\overline{\boldsymbol{V}}=0) \tag{8}$$

conserves the short-scale **enstrophy**

$$W \stackrel{\text{def}}{=} \frac{1}{2}\int dx\,[|\nabla\breve{\varphi}|^2+(\nabla^2\breve{\varphi})^2]=\sum_{k\text{ large}}\underbrace{\frac{1}{2}k^2(1+k^2)|\varphi_{\boldsymbol{k}}|^2}_{W_{\boldsymbol{k}}} \tag{9}$$

by multiplying Eq. (8) by $-\nabla^2\breve{\varphi}$, integrating over space, possibly integrating by parts, and ignoring any surface terms that arise. Begin with the $\overline{\boldsymbol{V}}$ term just in case you run out of time.

(e) [30 points] The constancy of $\sum_{\boldsymbol{k}} W_{\boldsymbol{k}}$ is analogous to the normalization integral of a probability density function (PDF) f: $\int_{-\infty}^{\infty}dx\,f(x)=\text{const.}\ (=1)$. Thus $W_{\boldsymbol{k}}\sim f(x)$ plays the role of a PDF for wave number $\boldsymbol{k}$. *Just accept this statement,* even if it seems unusual to you.

Because of the nonlinearity, turbulence arises, so the advection velocity $\overline{\boldsymbol{V}}$ becomes a random variable (denoted by tilde): $\overline{\boldsymbol{V}}\to\widetilde{\overline{\boldsymbol{V}}}$. Therefore the advection frequency, Eq. (5b), is also a random variable $\widetilde{\Omega}_{\boldsymbol{k}}$. Assume that it has zero mean, $\langle\widetilde{\Omega}_{\boldsymbol{k}}\rangle=0$, and autocorrelation time τ_{ac}. For definiteness, take

$$\langle\widetilde{\Omega}_{\boldsymbol{k}}(\boldsymbol{X},T+\tau)\widetilde{\Omega}_{\boldsymbol{k}}(\boldsymbol{X},T)\rangle=\langle\widetilde{\Omega}^2_{\boldsymbol{k}}(\boldsymbol{X},T)\rangle e^{-|\tau|/\tau_{\text{ac}}}. \tag{10}$$

That assumption turns the ray equation (7b) into the *random Langevin equation*

$$\frac{d\widetilde{\boldsymbol{k}}}{dT}=-\nabla\widetilde{\Omega}_{\boldsymbol{k}}(\boldsymbol{X},T). \tag{11}$$

(analogous to the one for classical Brownian motion $\dot{\widetilde{x}}=\widetilde{v}$). ***From the Langevin equation (11), show how to construct the Fokker–Planck diffusion equation for $W_{\boldsymbol{k}}$,***

$$\frac{\partial W_{\boldsymbol{k}}}{\partial T}=\frac{\partial}{\partial\boldsymbol{k}}\cdot\mathsf{D}_{\boldsymbol{k}}\cdot\frac{\partial W_{\boldsymbol{k}}}{\partial\boldsymbol{k}}. \tag{12}$$

- Consider only the second Fokker–Planck moment $\langle\Delta\boldsymbol{k}\,\Delta\boldsymbol{k}\rangle$. (Assume that if you calculated the first moment, it would obey the usual relation that turns the Fokker–Planck equation into the standard form of the diffusion equation.)
- Briefly explain all steps. (But don't write out all details of, say, reducing double time integrals to single ones unless you really need to.)
- Average $\langle\Delta\boldsymbol{k}\,\Delta\boldsymbol{k}\rangle$ over both the statistics of $\widetilde{\Omega}_{\boldsymbol{k}}$ as well as $\boldsymbol{X}$. You should obtain a wave-number diffusion tensor

$$\mathsf{D}_{\boldsymbol{k}}=2\tau_{\text{ac}}\left(\frac{k^2}{1+k^2}\right)^2\sum_{q\text{ small}}k_y^2\left(\frac{q^2}{1+q^2}\right)W_{\boldsymbol{q}}\,(\widehat{\boldsymbol{q}}\,\widehat{\boldsymbol{q}}), \tag{13}$$

  where the Cartesian coordinate system is relative to $\boldsymbol{q}$, so that $k_x=\widehat{\boldsymbol{q}}\cdot\boldsymbol{k}$, $k_y=\widehat{\boldsymbol{z}}\cdot\widehat{\boldsymbol{q}}\times\boldsymbol{k}$.
- ***Hint:*** Parseval's formula is

$$\int_{-\infty}^{\infty}\frac{d\boldsymbol{X}}{V}A(\boldsymbol{X})B(\boldsymbol{X})\equiv\overline{A(\boldsymbol{X})B(\boldsymbol{X})}=\sum_{\boldsymbol{q}}A_{\boldsymbol{q}}B^*_{\boldsymbol{q}}. \tag{14}$$

Part II, Question 5

Diagnostics [15 points]

Describe how you would make a tokamak diagnostic for THREE of the following:

1. Electron temperature
2. Ion temperature
3. Plasma current
4. Impurity content
5. Fusion power

Part II, Question 6A

Density Measurements in Plasmas [15 points]

Explain how you could use the O-mode (ordinary mode) in the microwave frequency range to measure the line-averaged density of a quasi-neutral positron-electron plasma. Give an expression relating the electron density to a physical quantity you can directly measure.

Part II, Question 6B

Tokamak essay [15 points]

Describe how a tokamak works, in as much detail as time allows. What are the approximate values of the main parameters of a large tokamak like TFTR (e.g. magnetic field,plasma current, temperatures, densities, etc).