General Exam, Part I

May 2003

Q&A

Part I, Question 1

Experimental Quickie

15 points

A. A pair of careful student experimenters have measured the DC Paschen breakdown curves for helium and air. They find that the minimum voltage on the Paschen curve for helium is lower than that on the curve for air. Did they make a mistake? Explain your answer. (5 pts)

B. In designing a simple homodyne microwave interferometer for measuring time-dependent plasma density behavior, the experimenter has the freedom to choose the initial phase shift between the reference and "signal" microwave beams. What value should be chosen and why? Hint:: For a steady-state plasma - whose density and size cause less than a 1/2 a fringe shift - sketch the interferometer signal as a function initial phase. (5 pts)

C. Sketch a Rogowski loop, explain how it works, and state what it is used to measure. (5 points)

Q+A

Part I, Question 2

General Phenomena (40 points)

Consider a rotating, axisymmetric, cylindrical column of electrons (no positive ions present) confined in the radial direction by a uniform axial magnetic field $B_0\mathbf{e}_z$ where $B_0 = const$ is assumed for simplicity. Assume that the equilibrium ($\partial/\partial t = 0$) electron density profile $n(r)$ and transverse pressure profile $P_\perp(r)$ are specified by

$$n(r) = \widehat{n} = const.,$$
$$P_\perp(r) = \widehat{n}\widehat{T}_\perp(1 - r^2/r_p^2), \tag{1}$$

for $0 \leq r < r_p$, where r is the radial distance from the cylinder axis, and r_p is the radius of the plasma column. Here, $\widehat{n}$ and $\widehat{T}_\perp$ are constants, and $n(r) = 0$ and $P_\perp(r) = 0$ for $r > r_p$.

(a) (5 points) Determine the equilibrium radial self-electric field $E_r(r)$ produced by the net electron charge density in the column interior ($0 \leq r < r_p$).

(b) (15 points) Assume that the plasma column rotates about the cylinder axis with azimuthal flow velocity $V_\theta(r) = \omega_r r$, where $\omega_r = const$. What is the macroscopic equilibrium radial force balance equation for a fluid element rotating about the cylinder axis? Assume that $B_0 = const.$ (independent of r).

(c) (10 points) From the radial force balance condition obtained in Part(b), show that there are two allowed solutions for the equilibrium angular rotation velocity, $\omega_r = \omega_r^+$ or $\omega_r = \omega_r^-$, where

$$\omega_r^\pm = \frac{\omega_c}{2}\left\{1 \pm \left[1 - \left(\frac{2\omega_p^2}{\omega_c^2} + \frac{2r_L^2}{r_p^2}\right)\right]^{1/2}\right\}. \tag{2}$$

In Eq. (2), ω_p is the plasma frequency, ω_c is the cyclotron frequency, and $r_L = (4\widehat{T}_\perp/m\omega_c^2)^{1/2}$ is the effective Larmor radius.

(d) (5 points) Use Eq. (2) to determine the maximum electron density $\widehat{n}_{max}$ that can be radially confined for specified values of B_0, $\widehat{T}_\perp$ and r_p. What are the values of ω_r^+ and ω_r^- at this maximum density?

(e) (5 points) Consider the cold-plasma limit with $\widehat{T}_\perp \to 0$. Evaluate the self-electric field energy (per unit axial length), $U_E = 2\pi\int_0^{r_p} dr r|E_r^2|/8\pi$, and the rotational kinetic energy, $U_K = 2\pi\int_0^{r_p} dr r n m V_\theta^2/2$. Show that

$$U_K = \frac{1}{2}\frac{\omega_p^2 r_p^2}{c^2}\frac{\omega_r^2}{\omega_c^2}U_B,$$
$$U_E = \frac{1}{8}\frac{\omega_p^2 r_p^2}{c^2}\frac{\omega_p^2}{\omega_c^2}U_B, \tag{3}$$

where $U_B = 2\pi\int_0^{r_p} dr r B_0^2/8\pi = (1/8)B_0^2 r_p^2$ is the magnetic field energy per unit axial length. Note from Eq. (3) that U_E, $U_K \ll U_B$ provided $c/\omega_p \gg r_p$.

Q+A

Part I, Question 3

MHD Equilibrium and Stability (40 points)

a) An axisymmetric magnetic field can be written in the form $\mathbf{B} = \nabla\psi \times \nabla\phi + F\nabla\phi$. Show that if $\mathbf{B}$ satisfies the equilibrium equation, then F must be constant on flux surfaces, $F = F(\psi)$. (10 points)

b) Consider the Taylor series expansion of ψ in R and z about the magnetic axis. What can you say about the first order terms in the expansion? If the flux surfaces are up-down symmetric, what does that tell you about the second order terms in the expansion? (You can assume that the magnetic axis is at $z = 0$.) What can you then say about the shape of the flux surfaces near the magnetic axis? What is the condition for the flux surfaces to be circular near the magnetic axis? (10 points)

c) Now consider a field in cylindrical geometry, $\mathbf{B} = B_\theta(r)\hat{\theta} + B_z(r)\hat{z}$. What is the curvature of the field lines? (You may find it helpful to use some expressions from the Plasma Formulary.) (10 points)

d) In class, we derived an expression for δW containing the term $-\int_V (\xi_\perp \cdot \nabla p)(\xi_\perp^* \cdot \kappa) d3x$. What is the physical significance of this term? What form does this take in cylindrical geometry? What can you say about the contribution of this term to the MHD stability of a cylindrical plasma confinement configuration? (10 points)

Q+A

Part I, Question 4
Waves Long Problem
30 Points

Lower Hybrid Heating and Current Drive in Tokamaks

A lower hybrid current drive system has been built by PPPL in collaboration with MIT for the C-Mod tokamak. It will provide up to 3 MW of source power at 4.6 GHz to a typical C-Mod plasma (electron density ~ 1.5×10^{14} cm^{-3} and B_T ~ 5 T). The wave launcher is designed to excite waves on the outboard side of the plasma near the midplane with a toroidal wave number spectrum peaked at $n_{toroidal} \sim n_{//} = ck_{//}/\omega = 2.5$. Assume that the plasma is composed of only deuterium ions and electrons.

(a) What is the relative ordering of the ion cyclotron frequency, the electron cyclotron frequency, the plasma frequency and the source frequency for these plasmas? [5 points]

(b) Estimate the perpendicular wavelength for these waves, assuming that the cold, electrostatic dispersion relation is an adequate model. [15 points]

(c) In which direction (radial, toroidal, or poloidal) does the wave power primarily flow in the plasma? Justify your answer. [10 points]

Q&A

Part I, Question 5
Waves Quickie
20 points

Derive an approximate dispersion relation for Bernstein waves that propagate purely perpendicular to the equilibrium magnetic field in a homogenous, quasineutral isotropic hot plasma composed entirely of positrons and electrons.

Helpful information:

The hot plasma electrostatic dispersion relation can be written in the form:

$$k_\perp^2+k_{//}^2+\sum_s \frac{4\pi n_s q_s^2}{T_{//}}\left[1+\sum_n \frac{(\omega-k_{//}V-n\Omega)T_\perp+n\Omega T_{//}}{k_{//}w_{//}T_\perp}e^{-\lambda}I_n(\lambda)Z(\varsigma_n)\right]=0$$

Q+A

Part I, Question 6.
Math Quickie
[15 points]

Evaluate W(x) for large x:

$$W(x) = \int_0^\infty e^{-(t+x/t)} dt$$

Q + A

Part I, Question 7: Classical transport [20 points]

This question concerns temperature-related eigenmodes of a two-species plasma with $\boldsymbol{B} = 0$.

(a) **[5 points]** For a collision-dominated regime, write the two-fluid temperature equations for such a plasma. Indicate both the forms of the heat-flow vector $\boldsymbol{q}$ (including the flow-driven piece) and the heat-generation term Q.

(b) **[15 points]** The complete two-fluid system (including equations for density and flow velocity) has ten linear eigenmodes. It is asserted that one of those modes describes temperature perturbations that behave nontrivially even in the limit of infinite wavelength.

- Briefly justify that assertion.
- Write the temperature equations linearized around a constant background for infinite-wavelength perturbations and calculate the relevant eigenvalue and eigenvector.
- Physically, what does this eigenmode describe?

DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 13, 2003

9 a.m. - 1 p.m.

- Answer all problems. Problem 1 has a choice of A or B (answer one only).

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem ____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Problems for Part II, May 13, 2003

1)	Do **either** 1A or 1B	45 points
	1A - Computational Methods	
	1B – Experimental Methods	
	Part 1 – Probes	
	Part 2 - Diagnostics	
2)	Neoclassical Theory	20 points
3)	Plasma Diagnostics	20 points
4)	Irreversible Processes	45 points
5)	Math	30 points
6)	Single Particle Motion	20 points

Total - 180 points

Part II, Problem 1A
Computational Methods (45 points)

Do <u>either</u> this page <u>or</u> Problem 1B

IA. Analysis of a Finite Difference Equation:
Consider the scalar one dimensional equation in conservation form

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=0 \qquad (1)$$

Now consider the following (2-step) finite difference approximation to Eq. (1):

$$U_j^* = U_j^n - \frac{\delta t}{\delta x}\left[F_j^n - F_{j-1}^n\right] \qquad (2a)$$

$$U_j^{n+1} = \frac{1}{2}\left\{U_j^n + U_j^* - \frac{\delta t}{\delta x}\left[F_j^* - F_{j-1}^*\right] - \frac{\delta t}{\delta x}\left[F_j^n - 2F_{j-1}^n + F_{j-2}^n\right]\right\} \qquad (2b)$$

Here, we use the (standard) notation that $U_j^n = U(t^n, x_j)$, where $t^n = n\,\delta t$, and $x_j = j\,\delta x$.

1. (20 points) Apply the algorithm to the linearized form of Equation (1) where $\partial F(U)/\partial x = A \times \partial U/\partial x$ with A a constant. Show that this algorithm is second order accurate in time and space.

2. (20 points) Use Von Neuman Stability Analysis to calculate the range of values of δt for which the linearized method is stable.

3. (5 points) What are some possible advantages and disadvantages of using the method (2a,b) for solving Eq. (1), compared to other finite difference methods.

Part II, Question 1B
(Do either this page or problem 1A)

Experimental Methods [45 points total]

Probes [25 points] You are presented with two black boxes. In one there is a hydrogen plasma of cold ions and 2 eV electrons at a density of 10^{13} cm^{-3}. In the other there is a singly-ionized U238 plasma, with identical T_e, T_i, and n_e. However, the boxes are unlabeled - you don't know which is which.

Each box is fitted with a single-tip Langmuir probe with a tip diameter of 1 mm and a length of 5 mm. The probe has drive electronics which allow you to take an I-V trace, float the probe and measure its voltage, or heat the probe into strong emission, thereby measuring the plasma potential. The plasma inside the box is unmagnetized, the box is constructed of a conductor, and is grounded.

a) Describe at least TWO techniques which would allow you to use the probe systems to identify the plasmas. Estimate the accuracy of each technique.
b) Sketch the I-V (current-voltage) characteristics for the Langmuir probes on the two boxes. Indicate the regions of interest in the characteristics. Indicate at least approximate values for the intercepts of the characteristic with the axes.
c) Another researcher brings in a third box containing a plasma, fitted with an identical Langmuir probe. The plasma again has identical parameters, and is known to be either hydrogen, deuterium, or singly-ionized helium. You are, however, now informed that the *accuracy* of the electron temperature determination is ± 1 eV. Is it still possible to identify the ionic species of the plasma in the new box? Why? If the same uncertainty applied to the probe systems on the first two boxes, would it affect your identification of the ionic species in those boxes?

Plasma Diagnostics [20 points] Describe three different ways to measure ion temperature in plasmas. For each method, briefly describe the physical principal it is based on, and some of its relative advantages and disadvantages.

Part II, Question 2.
Neoclassical Theory [20 points]

Estimating the cross-field (radial) flux of particles due to the collisional relaxation of a plasma in an axisymmetric toroidal system requires knowledge of the radial particle velocity (V_r). Consider the usual model for a tokamak with a strong toroidal field and weak poloidal field, and allow for the presence of a parallel electric field (i.e., dominantly in the toroidal direction),

(a) If magnetic trapping of particles is ignored, what is V_r?

(b) For magnetically trapped particles, you can use the conservation of canonical angular momentum (P_ζ) to determine V_r. Give an expression for P_ζ in terms of the poloidal flux function ψ, and use it to show that over a magnetically-trapped particle period,

$$\frac{\partial \psi}{\partial t} = -\mathbf{V} \cdot \nabla \psi$$

(c) Use Faraday's Law together with your result from Part (b) to obtain an estimate for the radial velocity of the magnetically trapped particles.

Part II, Question 3
Plasma Diagnostics [20 points]

Consider an interferometer that operates at a radiation wavelength $\lambda = 10.6\ \mu m$ (CO_2 laser) in the presence of spurious vibrations of the optical components. To compensate for these vibrations, interferometry is performed simultaneously using the same optical components at wavelength $\lambda = 0.633\ \mu m$ (HeNe laser). The HeNe interferometer is affected much less than the CO_2 by the plasma phase shift, but still somewhat. If $\omega >> \omega_p$ for both wavelengths:

(a) Derive expression for the plasma line integrated density $\int n_e\, dl$ in terms of the phase shift ϕ_c and ϕ_h , of the two interferometers.

(b) If ϕ_h can be measured with an accuracy of +/- π, what uncertainty does this introduce into the plasma density measurement?

(c) Thus evaluate the fractional error in measuring a 1 m thick plasma of density $10^{20}\ m^{-3}$, assuming ϕ_c is measured exactly.

Part II, Question 4:
Irreversible Processes and Classical Transport
[45 points]

This question addresses the calculation of viscosity in a magnetized plasma ($\boldsymbol{B} = B\widehat{\boldsymbol{z}}$). Although the general problem is very tedious, here we will merely consider a simple special case.

Viscosity describes the transport of momentum. Consider the situation shown in Fig. 1, where the fluid velocity is assumed to have the form $\boldsymbol{u} = u_y(x)\widehat{\boldsymbol{y}}$.

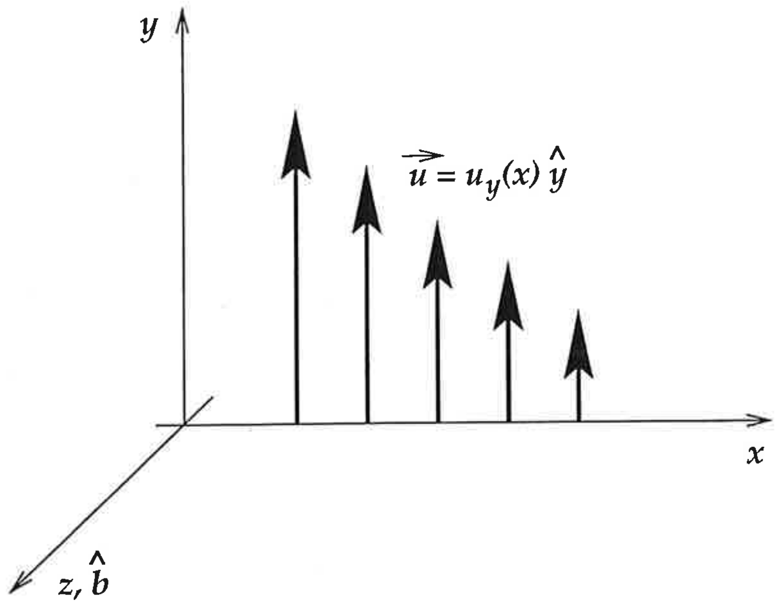

FIG. 1 A shear flow $\boldsymbol{u} = u_y(x)\widehat{\boldsymbol{y}}$ is assumed. The constant magnetic field is in the z direction.

Due to collisions, the momentum will diffuse, so one expects a law

$$nm\frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial}{\partial x} nm\mu \frac{\partial}{\partial x}\boldsymbol{u} \tag{1}$$

(a species label is omitted), or for constant background parameters

$$\frac{\partial u_y}{\partial t} = \mu \frac{\partial^2 u_y}{\partial x^2}. \tag{2}$$

On general principles, one expects that μ has the usual scaling for a cross-field classical transport coefficient, namely $\mu \sim \rho^2\nu$, where ρ is the gyroradius and ν is an appropriate collision frequency. (We assume $\nu/\omega_c \ll 1$.)

(a) [3 points] For $T_e = T_i$, how does the ratio μ_i/μ_e scale with mass? Which is larger, μ_e or μ_i?

(b) [8 points] In the theory of Brownian motion, it is demonstrated that an x-space particle diffusion coefficient D scales like $D \sim v^2\tau_{\rm ac}$ for some characteristic microscopic velocity v and appropriate correlation time $\tau_{\rm ac}$. This is also true for other transport coefficients like μ. For the magnetized μ described above, what is the effective $\tau_{\rm ac}$? (It is neither ν^{-1}, as it would be for parallel transport, nor ω_c^{-1}, the only other time scale in the problem.)

Assume that a certain correlation function $C(\tau)$ underlies μ in the same way that the velocity correlation function underlies D. Rederive your answer for $\tau_{\rm ac}$ by assuming that

$$C(\tau) = \cos(\omega_c\tau)e^{-\nu|\tau|}. \tag{3}$$

(Again, we are assuming $\nu/\omega_c \ll 1$.)

(c) [9 points] One way of estimating μ is to consider unbalanced microscopic momentum fluxes. See the diagram in Fig. 2.

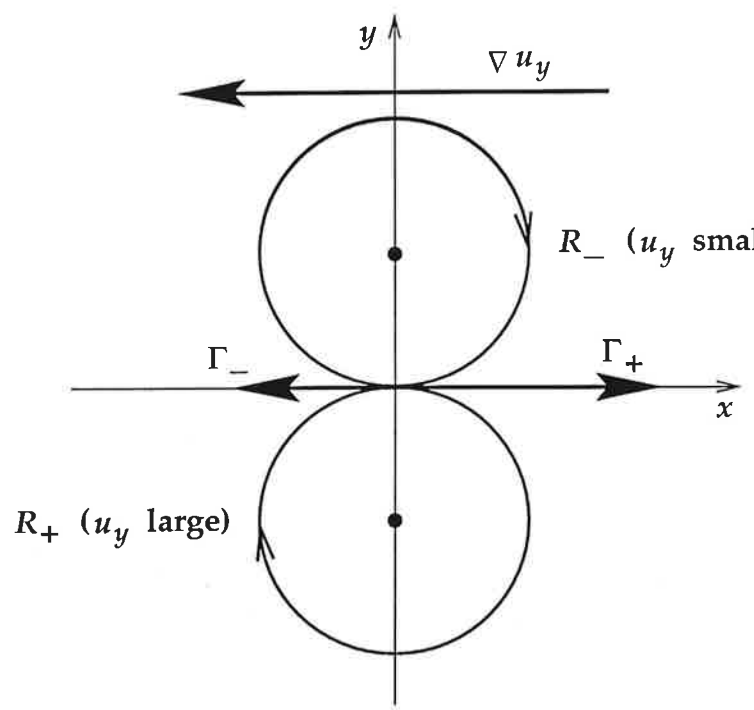

FIG. 2 Diagram for constructing a heuristic estimate of the x flux of y momentum. The gradient of the y-directed fluid velocity is in the $-x$ direction.

However, if one simply estimates the unbalanced x-directed flux of y-momentum density nmu_y (by transporting momentum from the regions R_+ and R_-), one gets the wrong answer.

- Show that such a calculation leads to $\mu \sim v_t^2\tau$ with the incorrect time $\tau = \omega_c^{-1}$.
- **Argue clearly why this estimate fails and give an improved argument that takes into account the result of part (b).**

(d) [25 points] The stress tensor is defined by

$$\Pi = \bar{n}m \int d\boldsymbol{v}\, \boldsymbol{v}\boldsymbol{v}\, f, \tag{4}$$

(in the frame of the moving flow) and at first order ($f_1 = \chi f_M$) the Chapman–Enskog correction equation is

$$(i\widehat{M} + \widehat{C})|\,\chi\,\rangle = -\tfrac{1}{2}v_t^{-2}|\,\boldsymbol{v}\boldsymbol{v} - \tfrac{1}{3}v^2\mathsf{I}\,\rangle : \mathsf{W} + \cdots, \tag{5}$$

where

$$\mathsf{W} \stackrel{\text{def}}{=} \boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T - \frac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathsf{I}, \tag{6a}$$

$$i\widehat{M} \stackrel{\text{def}}{=} \omega_c \boldsymbol{v} \times \widehat{\boldsymbol{b}} \cdot \frac{\partial}{\partial \boldsymbol{v}} = -\omega_c \frac{\partial}{\partial \phi} \tag{6b}$$

(ϕ is the cylindrical-polar angle in velocity space). For the special situation of this problem [$\boldsymbol{u} = u_y(x)\widehat{\boldsymbol{y}}$], discuss as completely as you can in the allotted time the quantitative calculation of μ.

- **[11 points]** First find a general expression for μ as a (scalar) matrix element in v space.
- **[2 points]** Do you expect $\widehat{C}_{ee}$ to contribute to μ_e? Why or why not?
- **[12 points]** Show how to calculate the ν_{ei} contribution to μ_e. Do not worry about numerical coefficients, but do show what integrals need to be done.

Hints:
(a) For two (possibly noncommuting) operators A and B,

$$(A + \epsilon B)^{-1} = A^{-1} - \epsilon A^{-1} B A^{-1} + O(\epsilon^2). \tag{7}$$

(b) The first few spherical harmonics Y_l^m are

$$Y_0^0 \propto 1, \tag{8a}$$
$$Y_1^0 \propto \cos\theta, \tag{8b}$$
$$Y_1^1 \propto \sin\theta e^{i\phi}, \tag{8c}$$
$$Y_2^0 \propto -1 + 3\cos^2\theta, \tag{8d}$$
$$Y_2^1 \propto \sin\theta\cos\theta e^{i\phi}, \tag{8e}$$
$$Y_2^2 \propto \sin^2\theta e^{2i\phi}. \tag{8f}$$

Part II, Question 5.
Math Problem
[30 points]

Consider the eigenvalue problem $y'' + (E - x^2)y = 0$ with $y(a) = y(-a) = 0$, $E > 0$.

(5 points) Sketch the Stokes plot for y.

(5 points) Find the values of E for $a \to \infty$.

(20 points) Carry out the analysis for the lowest value of E when a is finite. (You can shorten the work by requiring the solution to be even in x and doing the analysis only for $x > 0$.) Use Stokes constants for isolated singularities. Do not solve completely, but find the necessary equation to solve for E, and show that it reduces to the previous case for $a \to \infty$.

Part II, Question 6
Single-particle-motion quickie
20 points

Consider a current-carrying loop of wire (of radius **a**) and three identical (mass **m** and energy **E**) charged particles moving in its magnetic field. Initially the particles are placed exactly in the z = 0 plane and have zero z-directed velocity. Particles 1 and 3 are initially on the same field line; particle 2's gryo-orbit in centered on the z axis. All three have gyroradii much smaller than **a**.

a) For all three particles, sketch the orbit shapes viewed from above. The sketches should span a time longer than 3 gyro-periods for each particle.

b) Sketch ϕ vs t for all three particles. Give algebraic expressions for $\phi(t)$ for each.

c) If the current in the wire increases slowly with time, qualitatively what happens to the orbits for particles 2 and 3? Include comments on the gyro motion, the drift motion and the particle energy.

d) Returning to the case where the current in the wire is constant in time, qualitatively what happens to the orbits if the particles' initial positions are displaced a small distance from z = 0?