

DEPARTMENT OF ASTROPHYSICAL SCIENCES  
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 17, 2004

9 a.m. - 1 p.m.

- Answer all problems.
- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.
- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem \_\_\_\_ " and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Problems for Part I, May 17, 2004

1)	Experimental Plasma Physics Diagnostics - long	25 points
2)	Experimental Plasma Physics Diagnostics – quickie	10 points
3)	Neoclassical theory	60 points
4)	Kinetic Waves Quickie	15 points
5)	Waves Quickie	20 points
6)	MHD Quickie	15 points
7)	General Phenomena	35 points

Total - 180 points

**Part I, Question 1**  
**Long experimental plasma physics question**  
**25 points**

**I) (25 points)** Consider an interferometer that operates at a radiation wavelength  $\lambda_c = 10.6 \mu\text{m}$  ( $\text{CO}_2$  laser) in the presence of spurious vibrations of the optical components. To compensate for these vibrations, interferometry is performed simultaneously using the same optical components at  $\lambda_h = 0.633 \mu\text{m}$  (HeNe laser). The HeNe interferometer is affected much less than the  $\text{CO}_2$  by the plasma phase shift, but still somewhat. If  $\omega \gg \omega_p$  for both wavelengths:

- (a) Derive an expression for plasma density  $\int n_e \cdot dl$  in terms of the phase shifts  $\phi_c$  and  $\phi_h$  of  $\text{CO}_2$  and HeNe interferometers.
- (b) If  $\phi_h$  can be measured to an accuracy of  $\pm \pi$ , what uncertainty does this introduce into the plasma density measurement?
- (c) Evaluate the fractional error in measuring a 1 m thick plasma of density  $10^{20} \text{ m}^{-3}$ , assuming  $\phi_c$  is measured exactly.

Hint:

$$\phi^{\text{plasma}} = \frac{e^2 \lambda}{4\pi c^2 m_e \epsilon_0} \int n_e \cdot dl$$

**Part I, Question 2**  
**Experimental plasma physics quickie**

**II) (10 points)** You have been asked to design an experiment for a space shuttle mission. The experiment involves two shuttles in very close orbit connected by a long insulated cable. A differential voltage bias will be applied between the shuttles and the current collected will be monitored as a function of the applied bias.

The mission will be flown at 200 km altitude, near the lower boundary of the F layer, where the local electron density is expected to be between  $10^5$  and  $10^6 \text{ cm}^{-3}$ . Since the plasma in this region is primarily generated by photoionization, electrons of 1 – 2 eV are expected, and the ionic species is predominantly singly ionized cold oxygen. The earth's magnetic field can be neglected (by declaration). The conducting (collection) area of each shuttle is  $20 \text{ m}^2$ .

- (a) Calculate the voltage and current requirements for the power supply which provides the bias.
- (b) Sketch the current-voltage characteristic which you would expect if the electron density were  $10^6 \text{ cm}^{-3}$ , and the electron temperature 2 eV.

Part I, Question 3

60 points

NEOCLASSICAL THEORY PROBLEM

a) Calculate the electrical resistivity of a hydrogen plasma in the Lorentz gas approximation for the simplest case where the E-field is directed parallel to a straight, uniform B-field. Consider the collision operator (in spherical polar coordinates,  $v$ ,  $\theta$ ,  $\phi$ ) to be given by:

$$C(f_e) = \nu_{ei} \left( \frac{v_{Te}}{v} \right)^3 \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} f_e \right),$$

with  $f_e$  being the electron distribution function,  $v_{Te}$  being the electron thermal velocity, and  $\nu_{ei}$  being the usual electron-ion Coulomb collision frequency.

Note:  $v_{\parallel} = v \cos \theta$ ,

$$\int d^3v = 2\pi \int_0^{\infty} dv v^2 \int_0^{\pi} d\theta \sin \theta,$$

$F_{Me} = (2n_0/\pi^{3/2} v_{Te}^3) \exp(-v/v_{Te})^2$ , and

$$\int_0^{\infty} d\bar{v} \bar{v}^7 \exp(-\bar{v})^2 = 3$$

with  $\bar{v} \equiv v/v_{Te}$ .

b) State (without proof) how this parallel resistivity is modified in the “banana” regime of a tokamak plasma and give a dimensionless measure of the collisionality characterizing this regime.

### Part I, Question 4

#### “Kinetic Waves” Quickie 15 points

Consider a hot, uniform, unmagnetized plasma in which both the electrons and ions are isotropic, non-drifting Maxwellians. Assume that the plasma is sufficiently dense so that  $\omega_{ps}^2 / \omega^2 \gg 1$ . Derive the dispersion relation for ion acoustic waves, assuming that  $T_e \gg T_i$ .

Useful information:

$$\chi_s = \frac{2\omega_p^2}{k^2 v_t^2} \left\{ 1 + \frac{\omega - k V_D}{k v_t} Z(\zeta) \right\}$$

$$Z(\zeta \gg 1) \sim i \sigma \sqrt{\pi} e^{-\zeta^2} - \frac{1}{\zeta} \left[ 1 + \frac{1}{2\zeta^2} + \dots \right] \quad \text{where } \sigma = \begin{cases} 0 & \text{if } y > |x|^{-1} \\ 1 & \text{if } y < |x|^{-1} \\ 2 & \text{if } y < -|x|^{-1} \end{cases}$$

$$Z(\zeta \ll 1) \sim i \sqrt{\pi} e^{-\zeta^2} - 2\zeta \left[ 1 - \frac{2\zeta^2}{3} + \dots \right]$$

## Part I, Question 5

### Waves “Quickie”

20 points

The plasma in the ionosphere at 200 km above the earth's surface can be modeled by a quasineutral mix of singly ionized hydrogen, helium and oxygen gases plus electrons. Consider the propagation of electromagnetic waves along the earth's magnetic field in this region of the ionosphere. Sketch the dispersion relations for the two possible modes as a function of wave frequency normalized to the hydrogen cyclotron frequency (i.e., plot  $n^2$  vs.  $\omega/\Omega_H$ ). (Assume that the plasma density and earth's magnetic field are roughly constant in the region under consideration.)

**Part I, Question 6**  
**15 points**  
**MHD quickie**

The potential energy for the linearized MHD equations,  $\delta W$ , can be written in the form

$$\delta W = \frac{1}{2} \int_V [\gamma p |\nabla \cdot \xi|^2 - \xi^* \cdot \nabla (\xi \cdot \nabla p) + |Q|^2 / \mu - \xi^* \cdot \mathbf{j} \times \mathbf{Q}] d^3x,$$

where

$$\mathbf{Q} \equiv \nabla \times (\xi \times \mathbf{B}).$$

The effect of gravity can be incorporated in the MHD equations by adding a term  $\rho \mathbf{g}$  to the force in the momentum equation, where  $\mathbf{g}$  is a constant vector (independent of  $\mathbf{x}$  and  $t$ ) and  $\rho$  is the mass density of the plasma. Derive the corresponding modification in  $\delta W$  to include the effect of gravity in the energy principle.



**General Phenomena**

1. Consider the relativistic motion of a test electron (charge  $= -e$ , and rest mass  $= m$ ) in the cylindrically symmetric equilibrium configuration with crossed electric and magnetic fields,  $E_r(r)\hat{e}_r$  and  $B_\theta(r)\hat{e}_\theta + B_z(r)\hat{e}_z$ . Here,  $r = (x^2 + y^2)^{1/2}$  is the radial distance from the axis of symmetry, and

$$E_r(r) = -\frac{\partial}{\partial r}\phi(r) , \quad B_z(r) = \frac{1}{r}\frac{\partial}{\partial r}[rA_\theta(r)] , \quad B_\theta(r) = -\frac{\partial}{\partial r}A_z(r) , \quad (1)$$

where  $\phi(r)$  is the electrostatic potential, and  $A_\theta(r)$  and  $A_z(r)$  are the azimuthal and axial components of the vector potential. In cylindrical polar coordinates, the total energy ( $H$ ), canonical angular momentum ( $P_\theta$ ), and axial canonical momentum ( $P_z$ ) of the test electron can be expressed as

$$H = (m^2c^4 + c^2p_r^2 + c^2p_\theta^2 + c^2p_z^2)^{1/2} - e\phi(r) , \quad (2)$$

$$P_\theta = r[p_\theta - \frac{e}{c}A_\theta(r)] , \quad (3)$$

$$P_z = p_z - \frac{e}{c}A_z(r) . \quad (4)$$

Here,  $\mathbf{p} = \gamma m \mathbf{v}$  is the mechanical momentum,  $\gamma = (1 + \mathbf{p}^2/m^2c^2)^{1/2}$  is the kinematic mass factor, and  $v_r = dr/dt$ ,  $v_\theta = r d\theta/dt$  and  $v_z = dz/dt$  are the velocity components in cylindrical polar coordinates.

- (a) The equation of motion for the test electron is

$$\frac{d\mathbf{p}}{dt} = -e \left[ E_r \hat{e}_r + \frac{1}{c} \mathbf{v} \times (B_\theta \hat{e}_\theta + B_z \hat{e}_z) \right] . \quad (5)$$

Show directly from Eqs. (1)–(5) that

$$\frac{dH}{dt} = 0 , \quad \frac{dP_\theta}{dt} = 0 , \quad \text{and} \quad \frac{dP_z}{dt} = 0 . \quad (6)$$

That is,  $H$ ,  $P_\theta$  and  $P_z$  are exact single-particle constants of the motion in the cylindrically symmetric field configuration described by Eq. (1).

- (b) Assume that the radial electric field  $E_r(r)$  and azimuthal self-magnetic field  $B_\theta(r)$  are generated by an intense beam of electrons with step-function density profile  $n_b(r)$ , and axial current density profile  $J_{zb}(r)$  given by

$$n_b(r) = \begin{cases} \hat{n}_b = \text{const.} , & 0 \leq r < r_b , \\ 0 , & r > r_b , \end{cases} \quad (7)$$

and

$$J_{zb}(r) = \begin{cases} -eV_b\hat{n}_b = \text{const.} , & 0 \leq r < r_b , \\ 0 , & r > r_b . \end{cases} \quad (8)$$

Solve the steady-state Maxwell equations for  $\phi(r)$  and  $A_z(r)$ , and show that the single-particle constants of the motion  $H$  and  $P_z$  defined in Eqs. (2) and (4) can be expressed as

$$H = (m^2c^4 + c^2p_r^2 + c^2p_\theta^2 + c^2p_z^2)^{1/2} - \frac{1}{4}m\omega_{pb}^2r^2 = \text{const.} \quad (9)$$

$$P_z = p_z - \frac{1}{4}mV_b\frac{\omega_{pb}^2r^2}{c^2} = \text{const.} \quad (10)$$

for test particle motion inside the beam ( $0 \leq r < r_b$ ). Here,  $\omega_{pb} = (4\pi\hat{n}_be^2/m)^{1/2}$  is the nonrelativistic plasma frequency.

### Useful Relations in Cylindrical Coordinates $(r, \theta, z)$

$$\left(\frac{d}{dt}\mathbf{p}\right)_r = \gamma m \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] + m \frac{dr}{dt} \frac{d\gamma}{dt} ,$$

$$\left(\frac{d}{dt}\mathbf{p}\right)_\theta = \gamma m \left[ r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right] + mr \frac{d\theta}{dt} \frac{d\gamma}{dt} ,$$

$$\left(\frac{d}{dt}\mathbf{p}\right)_z = \gamma m \frac{d^2z}{dt^2} + m \frac{dz}{dt} \frac{d\gamma}{dt} ,$$

$$\nabla^2 f(r) = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} f(r) .$$

DEPARTMENT OF ASTROPHYSICAL SCIENCES  
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 18, 2004

9 a.m. - 1 p.m.

- Answer all problems.
- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.
- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem \_\_\_\_ " and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Problems for Part II, May 18, 2004

- |    |                                |           |
|----|--------------------------------|-----------|
| 1) | Basic Plasma quickie           | 15 points |
|    | Do question A plus B or C      |           |
| 2) | Irreversible Processes         | 50 points |
| 3) | Laboratory in Plasma Physics   | 25 points |
|    | Do question A, B, or C         |           |
| 4) | Math                           | 50 points |
| 5) | General Plasmas - Planar Diode | 40 points |

Total - 180 points

**Part II, Question 1**  
**15 points**  
**Basic plasma quickies**

**Do question A, plus B or C:**

A. (5 points) Write down approximate values for the ratio of the ion-ion collision frequency to the electron-electron collision frequency, and for the ratio of the electron-ion energy exchange rate to the electron-electron collision frequency.

Choose B or C (10 points):

B. Sketch trapped-particle orbits in a large aspect ratio tokamak. Estimate the ratio of the banana width to the gyroradius, and show how it scales with the aspect ratio of a tokamak.

C. Consider a cylinder of plasma of initial radius  $a_0$ , magnetic field  $\vec{B} = B_0 \hat{z}$  and pressure  $p_0$ . The plasma is then compressed to radius  $a_1$  on a time scale that is slow compared to the gyrofrequency but fast compared to the collision frequency. What is the resulting thermal energy density of the plasma?

## Part II, Question 2: Irreversible Processes [50 points]

In principle, one can learn something about the thermal conductivity of a plasma by performing a scattering experiment and examining the line width associated with the thermal conduction mode. In this question, you will be asked to discuss some theoretical aspects of this method.<sup>1</sup>

The basic assumptions are

- the plasma is in *thermal equilibrium*;
- the plasma is assumed to be a *one-component electron plasma* (with a smooth neutralizing ion background);
- The plasma is *unmagnetized*.

Recall that the light-scattering cross section is proportional to the autocorrelation function of the electron density  $\langle \delta n_e \delta n_e \rangle(\mathbf{k}, \omega)$ . For this one-component plasma, that is essentially the autocorrelation function  $C$  of the fluctuating charge density  $\rho$ , namely  $C(\mathbf{k}, \omega) \stackrel{\text{def}}{=} \langle \delta \rho \delta \rho \rangle(\mathbf{k}, \omega)$ . For high-frequency, short-wavelength modes, it is possible to evaluate  $C(\mathbf{k}, \omega)$  from Rostoker's Test Particle Superposition Principle. However, thermal conduction is a low-frequency, long-wavelength (hydrodynamic) process, for which the Superposition Principle may not be applicable. Fortunately, in thermal equilibrium one has available the Fluctuation-Dissipation Theorem

$$C(\mathbf{k}, \omega) = - \left( \frac{k^2}{2\pi} \right) T \operatorname{Im} \left( \frac{1}{\omega \mathcal{D}(\mathbf{k}, \omega)} \right), \quad (1)$$

where  $\mathcal{D}(\mathbf{k}, \omega)$  is the electrostatic dielectric function.

*Read through the entire problem before you begin. The most tedious algebra is in part (a). Even if you don't complete that, however, you should be able to do something about the later parts.*

*(Problem continues on next page.)*

---

<sup>1</sup> Don't worry about possible experimental difficulties.

(a) [30 points] Find an expression for  $\mathcal{D}(\mathbf{k}, \omega)$  by inserting a test charge density into simple linearized fluid equations for the electrons.

- Only worry about longitudinal motions ( $\delta \mathbf{u} \parallel \mathbf{k}$ ).
- Include transport coefficients related to viscosity ( $\mu$ ), friction ( $\nu$ ), and thermal conductivity ( $\kappa$ ).

In particular, show that

$$\mathcal{D} = \frac{a}{a - 1}, \quad (2)$$

where

$$a = 1 - \frac{\omega(\omega + i\nu_\mu)}{\omega_p^2} + \kappa^2 \lambda_D^2 \left( 1 + \frac{(2/3)\omega}{\omega + i\nu_\kappa} \right) \quad (3)$$

with  $\nu_\mu \stackrel{\text{def}}{=} \mu k^2 + \nu$  and  $\nu_\kappa \stackrel{\text{def}}{=} \kappa k^2$ .

(b) [7 points] The dielectric function you derive should describe three normal modes. Let  $k^2 = O(\epsilon)$ . To *lowest order* in  $\epsilon$ , **what is the dispersion relation for each of those modes?** Identify the thermal conduction mode.

(c) [2 points] In the vicinity of a normal mode of (possibly complex) frequency  $\omega_0$ , **prove that**

$$\left( \frac{1}{\mathcal{D}(\mathbf{k}, \omega)} \right) \approx \frac{1}{(\omega - \omega_0) \mathcal{D}'(\mathbf{k}, \omega)|_{\omega_0}}, \quad (4)$$

where  $\mathcal{D}' \stackrel{\text{def}}{=} \partial \mathcal{D} / \partial \omega$ .

(d) [9 points] Use formula (4) to work out the FDT (1) for the thermal conduction mode. **Show that approximately**

$$C(\mathbf{k}, \omega) \approx 2\pi \Delta_k \delta(\omega); \quad (5)$$

give a formula for the intensity  $\Delta_k$ .

(e) [2 points] What is the true, nonzero width of the thermal conduction line in terms of the thermal conductivity coefficient  $\kappa$ ?

(f) [0 points] Be prepared to discuss the calculation of  $\kappa$ , just in case someone should ask you about it sometime.

**Part II, Question 3**  
**Laboratory in Plasma Physics**  
**25 points**

Chose to answer either A, B, or C. Options A and B are based on Grad Lab experiments which you may have done. Option C can be based on any other Grad Lab experiment.

A. DC Breakdown of Gases: Paschen Curve

1. Sketch the apparatus which was used in the Grad Lab to measure the DC breakdown voltage of a gas, i.e. to determine at what voltage a gas turns into a plasma. What did you actually measure ?
2. Describe in words the main physical parameters which determine this DC breakdown voltage for a given gas. Sketch an example of the Paschen curve with units appropriate to this experiment. Add to this description any equations you can remember which characterize this breakdown process.
3. What were some of the sources of error or uncertainties in this experiment ? How did you (or could you) quantify these ?
4. Suppose for a science demonstration you wanted to make a 1 meter long spark in room air. Approximately what DC voltage would you need? Show how you estimated this based on your lab experiment. What factors would influence this voltage in an actual demonstration ?

B. Microwave interferometry

1. Sketch the apparatus which was used in the Grad Lab to measure plasma density using microwave interferometry. Include as many of the components as you can remember and describe their functions.
2. Describe in words the main physical process by which we measured plasma density using microwave interferometry. Add to this description any equations you can remember which characterize this process.



3. What were some of the sources of error or uncertainties in the grad lab experiment? How did you (or could you) quantify these?
4. Suppose you have a toroidal plasma like that in the Alcator C-Mod tokamak with a central density of  $\approx 1 \times 10^{15} \text{ cm}^{-3}$  and an edge density of  $\approx 1 \times 10^{14} \text{ cm}^{-3}$ . What technique(s) would you use to measure the radial density profile using microwave interferometry? Assume toroidal and poloidal symmetry.

### C. Other grad lab experiments

Describe in as much detail as possible any other plasma experiment which you did in the Grad Lab. You can use questions similar to those above to organize your answer.

Part II, Question 4  
50 points

Generals      Math

Consider

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

- a) Find two local series solution about  $x = 0$ ,  $y_1(x)$  and  $y_2(x)$ .
- b) Find two possible asymptotic behaviors at  $x \rightarrow +\infty$
- c) Find two Fourier-Laplace integral solutions

$$y(x) = \int e^{ixt} f(t) dt$$

Hint - Use real and imaginary parts, and examine all possible contours.

- d) Identify the solutions in part (a) with those in part (c),  
ie find integral solutions for  $y_1(x)$  and  $y_2(x)$ .
- e) Find the behavior of the two solutions in part (a)  
at  $x \rightarrow +\infty$  including normalization.

Part II, Question 5  
40 points

*planar diode*

1. [40 points]

A planar diode in which the cathode is a copious emitter of electrons is limited in the amount of current it carries because of space-charge effects. Assuming divergence-free steady-state flow in one dimension, one can derive the important “3/2” law for space-charge limited diodes, namely, that the maximum current is proportional to the voltage to the 3/2 power. This “3/2” law works for 1-D flow of a non-neutral stream of ions or electrons.

In this problem, suppose a small variation: Suppose instead that a source of ions situated at  $x = 0$  emits ions in the  $x$  direction at speed  $v_0$  rather than at rest. Suppose further that there is a voltage drop of  $\phi_0$  between  $x = 0$  and  $x = L$ . The ions are extracted at  $x = L$ , and the question is what is the maximum steady state ion current  $J$  that can be extracted as a function of  $v_0$ ,  $\phi_0$ , and  $L$ , as well as ion charge  $q$  and mass  $m$ .

- (a) (5 pts) For a steady state flow of ions beginning with velocity  $v_0$  at  $x = 0$ , sketch the potential as a function of  $x$ . [You can do this step now based on physical intuition, or you can solve exactly and come back to this after you do part (d).]  
(b) (15 pts) Show that the potential must be of the form:

$$\left[ \frac{d\phi(x)}{dx} \right]^2 = \alpha [\phi_M - \phi(x)]^{1/2} + \beta,$$

where  $\alpha$  is a constant which is function of the current  $J$ , where  $\phi_M$  is a constant which is a function of  $\phi_0$  and  $v_0$ , and where  $\beta$  is a constant of integration. Determine  $\alpha$  and  $\phi_M$ .

Hint: a simple identity you might find useful is:

$$\frac{d\phi(x)}{dx} \frac{d^2\phi(x)}{dx^2} = \frac{d}{dx} \left\{ \frac{1}{2} \left[ \frac{d\phi(x)}{dx} \right]^2 \right\}.$$

- (c) (5 pts) Argue from physical principles that the maximum steady state current will be extracted when  $\beta$  is set to zero.  
(d) (15 pts) Derive an expression for the saturation current  $J$ . Show that, in the limit  $v_0 \rightarrow 0$ , the saturation current obeys the familiar “3/2” scaling law.