DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 21, 2007

9 a.m. - 1 p.m.

- Answer all problems.
  Problem 6 has a choice of (A) or (B). Answer only one.

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate <u>specifically</u> how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Contents (Part I, May 21, 2007)

Part I, Question 1:
Experimental Methods
[20 points]

(a) [10 points] What plasma characteristics can you measure with a single Langmuir probe if (i) it is biased at a constant potential (referenced to ground), or (ii) if the voltage bias is swept? Over what voltage range should it be swept (in terms of the plasma parameters), and why? What effect would a small ($n \ll n_{T,e}$), fast ($v \gg v_{T,e}$) electron population have on the probe characteristics? (Here the subscript T refers to the bulk or thermal electrons.)

(b) [5 points] What can you measure with a double Langmuir probe? What are the advantages of a double probe (if any) over a single probe?

(c) [5 points] What can you measure with a triple Langmuir probe? What assumptions are made about the plasma? What should the bias voltages for the three tips be?

Part I, Question 2:
General Phenomena
[35 points]

An axisymmetric ($\partial/\partial\theta = 0$) one-component plasma column consisting only of electrons (charge $= -e$, mass $= m$) is confined radially by a uniform axial magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ with $B_0 = \text{const}$. For an infinitely-long plasma column in thermal equilibrium with $\partial/\partial z = 0$, the electron distribution function is given by

$$f(r, \boldsymbol{p}) = \frac{\hat{n}}{(2\pi m k_B T)^{3/2}} \exp\left(-\frac{(H - \Omega P_\Theta)}{k_B T}\right). \tag{1}$$

Here $\hat{n}$ and Ω are constants, $T = \text{const}$ is the temperature, and H and P_Θ are the single-particle constants of the motion defined by

$$H = \frac{1}{2m}(p_r^2 + p_\theta^2 + p_z^2) - e\phi(r), \tag{2}$$

$$P_\Theta = r\left(p_\theta - \frac{1}{2}mr\omega_c\right), \tag{3}$$

where $\omega_c \overset{\text{def}}{=} eB_0/mc$ is the electron cyclotron frequency, $r \overset{\text{def}}{=} (x^2 + y^2)^{1/2}$ is the radial distance from the axis of symmetry, and $m\boldsymbol{v} = \boldsymbol{p} = (p_r, p_\theta, p_z)$ is the particle momentum in cylindrical polar coordinates. The equilibrium electrostatic potential $\phi(r)$ in Eqs. (1) and (2) is determined self-consistently from Poisson's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(r)}{\partial r}\right) = 4\pi e n(r), \tag{4}$$

where $n(r) = \int d^3p\, f(r, \boldsymbol{p})$ is the electron density profile.

(a) [**7 points**] Determine a closed expression for the average electron flow velocity $\boldsymbol{V}(\boldsymbol{x}) = (\int d^3p\, \boldsymbol{v} f)/(\int d^3p\, f)$ for the choice of distribution function in Eq. (1). What is the significance of the constant Ω?

(b) [**7 points**] Show from Eqs. (1)–(3) that the electron density profile can be expressed as

$$n(r) = \hat{n}\exp\left(-\frac{\Psi(r)}{k_B T}\right), \tag{5}$$

where

$$\Psi(r) \overset{\text{def}}{=} \frac{1}{2}mr^2(\Omega\omega_c - \Omega^2) - e\phi(r). \tag{6}$$

(Problem continues on next page.)

(c) [7 points] Without loss of generality, take $\phi(r{=}0) = 0$ in Eqs. (5) and (6). Then the constant $\widehat{n} = n(r{=}0)$ can be identified with the on-axis density at $r = 0$. Near $r = 0$, show from Eqs. (4)–(6) that the solution for $\phi(r)$ is given approximately by

$$\phi(r) \approx \frac{1}{4}\frac{m}{e}\omega_p^2 r^2, \tag{7}$$

where $\omega_p^2 \stackrel{\text{def}}{=} 4\pi\widehat{n}e^2/m$.

(d) [7 points] Show that the condition for $n(r)$ to be a monotonically decreasing function of r near $r = 0$ is

$$\Omega\omega_c - \Omega^2 - \frac{1}{2}\omega_p^2 > 0. \tag{8}$$

In fact, the inequality in Eq. (8) assures that the density profile $n(r)$ is a monotonically decreasing function of r for all r, corresponding to a radially confined equilibrium with $n(r \to \infty) = 0$.

(e) [7 points] For specified values of ω_p and ω_c,

- What is the maximum value of Ω consistent with a radially confined equilibrium?
- What is the minimum value of Ω?
- What is the largest value of ω_p^2/ω_c^2 for which the equilibrium is radially confined?
- What is the corresponding value of Ω/ω_c?

Part I, Question 3: MHD [15 points]

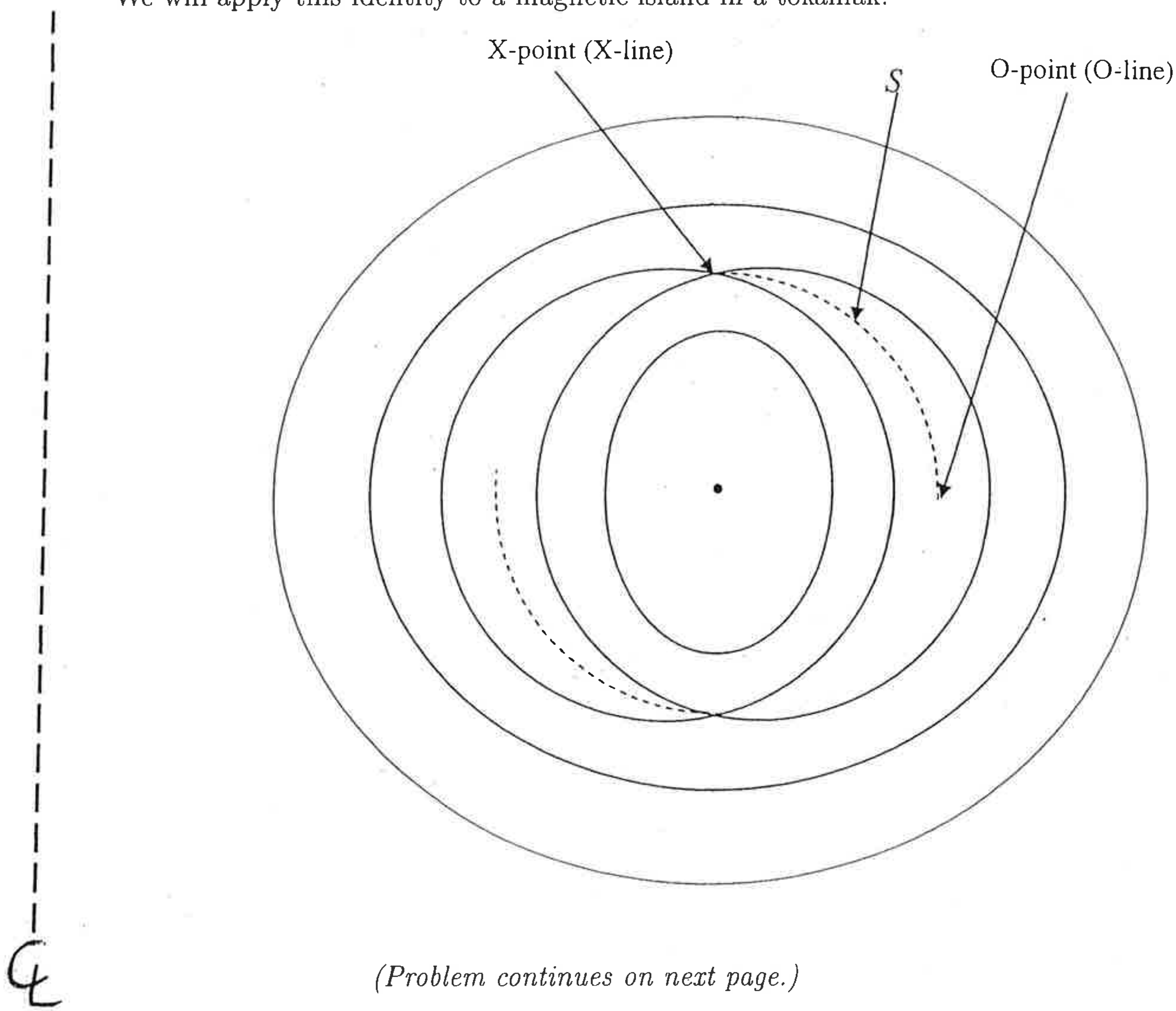
Given any two-dimensional surface S in a plasma, the magnetic flux through the surface is defined as

$$\psi \stackrel{\text{def}}{=} \int_S d^2x\, \widehat{\boldsymbol{n}} \cdot \boldsymbol{B}, \tag{1}$$

where the integration is taken over the surface S and $\widehat{\boldsymbol{n}}$ is the unit vector normal to the surface. If the surface moves with the plasma, then

$$\frac{d\psi}{dt} = \int_S d^2x\, \widehat{\boldsymbol{n}} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \oint_{\text{edge}} \boldsymbol{B} \cdot \boldsymbol{v} \times d\boldsymbol{l}, \tag{2}$$

where $\boldsymbol{v}$ is the local fluid velocity, $d\boldsymbol{l}$ is an element of length along the edge of the surface, and the second integral is taken around the edge of the surface. We will apply this identity to a magnetic island in a tokamak.



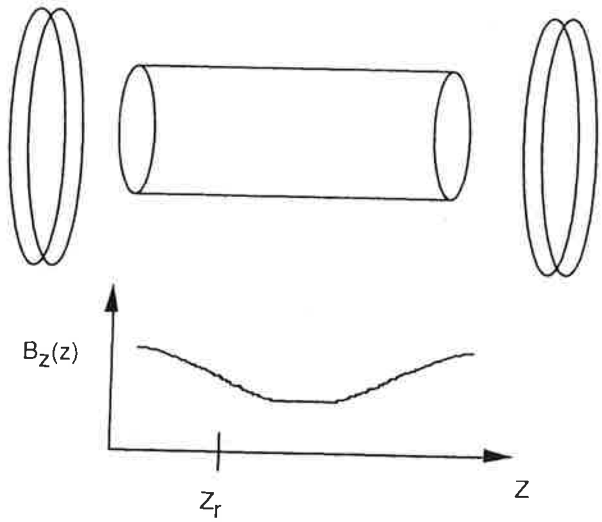
(Problem continues on next page.)

(a) [10 points] The figure on the previous page shows a Poincaré section of flux surfaces in a tokamak with a magnetic island. The surface S, indicated by a dashed line in the figure, connects the O-line and the X-line of the island. Using the resistive MHD equations, obtain an expression for the time derivative of the flux through this surface in terms of the resistivity and current density at the O-line and the X-line. What is $d\psi/dt$ in the limit that the resistivity goes to zero?

(b) [5 points] Suppose that the pressure gradient is flattened in the island, removing the bootstrap contribution to the current in the island, so that the current density along the O-line decreases. What effect will this have on the island?

Part I, Question 4:
Waves
[35 points]

A cylindrical plasma column is confined by an axial magnetic field $\boldsymbol{B} = B_z(z)\hat{\boldsymbol{z}}$ created by a pair of Helmholtz coils, as sketched below. (For simplicity, we ignore any radial dependence of $\boldsymbol{B}$.) The quasineutral plasma is composed of only electrons and deuterium ions.



A wave with a frequency ω_0 comparable to the magnitude of the deuterium ion cyclotron frequency is detected propagating only along the magnetic field towards the center of the plasma cell. The wave encounters a resonance with the ions at $z = z_r$.

Note: See possibly useful mathematical information at the conclusion of the question.

(a) [5 points] In the cold-plasma limit, what is the dispersion relation governing the propagation of this wave? [Assume that the WKB approximation is valid and that $|(L_z/B_z)dB_z/dz| \gg 1$, where L_z is the length of the plasma column.]

(b) [5 points] If the wave encounters a resonance at $z = z_r$, from where in the plasma could the wave have been launched? Provide a rough sketch indicating a location where the wave could have been launched and justify your answer.

(c) [5 points] What is the polarization of this wave in the cold-plasma limit? Does nonzero temperature change the wave polarization? Justify your answer.

(d) [5 points] Now assume that the plasma is warm with $T_D = T_e$. Identify all possible linear damping mechanisms in this plasma.

(e) [15 points] Derive the dispersion relation for these waves, retaining the lowest-order finite-temperature corrections and restricting attention to locations away from the exact resonance.

(Problem continues on next page.)

Hint: The following information may (or may not) be useful:

- For parallel propagation (no drifts, isotropic Maxwellians):

$$\boldsymbol{\chi}_s = \begin{pmatrix} \chi_\perp & -i\chi_x & 0 \\ i\chi_x & \chi_\perp & 0 \\ 0 & 0 & \chi_{zz} \end{pmatrix}_s , \tag{1}$$

where

$$\chi_{\perp,s} = \frac{1}{2}\left(\frac{\omega_{p,s}^2}{\omega^2}\right)\left(\frac{\omega}{k_\| v_{t,s}}\right)[Z(\zeta_{-1}) + Z(\zeta_1)]_s, \tag{2a}$$

$$\chi_{x,s} = \frac{1}{2}\left(\frac{\omega_{p,s}^2}{\omega^2}\right)\left(\frac{\omega}{k_\| v_{t,s}}\right)[Z(\zeta_{-1}) - Z(\zeta_1)]_s, \tag{2b}$$

$$\chi_{zz,s} = 2\left(\frac{\omega_{p,s}^2}{k_\|^2 v_{t,s}^2}\right)\left[1 + \left(\frac{\omega}{k_\| v_t}\right) Z(\zeta_0)\right]_s , \tag{2c}$$

$$\zeta_{n,s} \stackrel{\text{def}}{=} \frac{\omega - n\Omega_s}{k_\| v_{t,s}}. \tag{2d}$$

- Large-argument limit of the Z function:

$$Z(\zeta \gg 1) \sim i\sigma\sqrt{\pi}e^{-\zeta^2} - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \cdots\right), \tag{3}$$

where

$$\sigma \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } y > |x|^{-1}, \\ 1 & \text{if } y < |x|^{-1}, \\ 2 & \text{if } y < -|x|^{-1}. \end{cases} \tag{4}$$

- Small-argument limit of the Z-function:

$$Z(\zeta \ll 1) \sim i\sqrt{\pi}e^{-\zeta^2} - 2\zeta\left(1 - \frac{2}{3}\zeta^2 + \cdots\right). \tag{5}$$

Part I, Question 5:
Neoclassical Theory
[35 points]

This problem deals with the influence of trapped particles on transport in a toroidal system. In a tokamak plasma, the "neoclassical" dynamics in the long-mean-free-path "banana regime" can produce cross-field transport that is significantly larger than the classical diffusion rate.

(a) [20 points] For a simple model tokamak with circular cross section, use the conservation of canonical angular momentum to estimate the width of the banana orbit (for deeply trapped particles) in the banana regime.

(b) [10 points] Using the result from part (a), make a random-walk estimate of the diffusion coefficient and compare with classical diffusion.

(c) [5 points] Estimate the collision frequency at which the transition from the banana regime to the (higher collisionality) "plateau regime" occurs.

Do only one of the next two questions: either

- *Question 6a (Applied Mathematics), or*
- *Question 6b (Computational Methods).*

Part I, Question 6a:
Applied Mathematics
[40 points]

Optional: Do either this question or Question 6b (Computational Methods).

Consider

$$\frac{d^2y}{dz^2} + z^2 y = 1. \tag{1}$$

(a) [5 points] Find all possible asymptotic behaviors for $z \to 0$.

(b) [10 points] Find the leading asymptotic behavior for $z \to +\infty$ of (i) the particular solution, and (ii) the two solutions of the homogeneous equation.

(c) [25 points] Find an integral representation of the solution of the inhomogeneous equation.

Part I, Question 6b: Computational Methods [40 points]

Optional: Do either this question or Question 6a (Applied Mathematics).

In this problem you are asked to analyze the **Wendroff method**.

Consider the following simple 1D advection equation for the scalar quantity $U(x,t)$:

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0. \tag{1}$$

Here a is a constant, and x and t refer to space and time. Consider a standard space-time grid where the unknown $U_j^n = U(x_j, t^n)$ is defined at integer values of j and n so that $x_j = j\,\delta x$ and $t^n = n\,\delta t$. The Wendroff algorithm finite-differences this equation as

$$\frac{U_{j+1/2}^{n+1} - U_{j+1/2}^{n}}{\delta t} + a\left(\frac{U_{j+1}^{n+1/2} - U_{j}^{n+1/2}}{\delta x}\right) = 0. \tag{2}$$

The half-integer subscripts in Eq. (2) refer to evaluating the quantity halfway between two grid points by simple averaging, i.e.,

$$U_{j+1/2}^{n} \stackrel{\text{def}}{=} \frac{1}{2}(U_j^n + U_{j+1}^n), \tag{3a}$$

$$U_j^{n+1/2} \stackrel{\text{def}}{=} \frac{1}{2}(U_j^n + U_j^{n+1}), \tag{3b}$$

etc.

(a) [**5 points**] By substituting Eqs. (3) and similar relations into Eq. (2), rewrite Eq. (2) so that it only involves integer space and time points.

(b) [**10 points**] Use von-Neumann stability analysis to investigate the numerical stability of the finite-difference equation obtained in part (a) as a function of the time step δt, the spatial increment δx, and the constant a.

(c) [**10 points**] What are the leading-order truncation errors of the Wendroff algorithm (2) as applied to Eq. (1)?

(d) [**10 points**] How would you solve the system of difference equations obtained in part (a)? How many arithmetic operations would be required to advance one step in time if there were N grid points?

(e) [**5 points**] What are some relative advantages of the Wendroff algorithm (2) compared to other methods for solving Eq. (1)?

DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 22, 2007

9 a.m. - 1 p.m.

- Answer all problems.

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

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Contents (Part II, May 22, 2007)

Part II, Question 1:
Collisions
[15 points]

ITER aims to produce 400 MW of fusion power with a plasma having core parameters $T \sim 20\,\mathrm{keV}$, $n_e \sim 10^{20}/\mathrm{m}^3$, $R \sim 6\,\mathrm{m}$, $a \sim 2\,\mathrm{m}$, and $B \sim 5\,\mathrm{T}$. One of the research questions is whether the pressure of the alpha particles will ever be so large as to drive instabilities.

Based on simple power-balance and drag-rate considerations, give a simple back-of-the-envelope estimate of the normalized alpha-particle pressure β_α.

Part II, Question 2: MHD [45 points]

In this problem, you will be asked to use the MHD energy principle to analyze the Raleigh–Taylor instability.

The effect of gravity can be incorporated into the MHD equations by adding a term $\rho \boldsymbol{g}$ to the force in the momentum equation, where $\boldsymbol{g}$ is a constant vector (independent of both $\boldsymbol{x}$ and t) and ρ is the mass density of the plasma. We will consider a plasma in slab geometry supported in equilibrium against a gravitational force $\rho \boldsymbol{g} = -\rho g \widehat{\boldsymbol{x}}$ by a magnetic field $\boldsymbol{B} = B(x)\widehat{\boldsymbol{z}}$, with pressure $p = 0$.

(a) [**8 points**] Express the current density in terms of the magnetic field. Give the equilibrium force balance relation between B, ρ, and g.

(b) [**37 points**] Including the effect of gravity, δW can be written in the form

$$\delta W = \frac{1}{2}\int_V d^3x\, [\gamma p |\boldsymbol{\nabla}\cdot\boldsymbol{\xi}|^2 - \boldsymbol{\xi}^*\cdot\boldsymbol{\nabla}(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p) + |Q|^2/\mu - \boldsymbol{\xi}^*\cdot\boldsymbol{j}\times\boldsymbol{Q} + \boldsymbol{\xi}^*\cdot\boldsymbol{g}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi})], \tag{1}$$

where

$$\boldsymbol{Q} \overset{\text{def}}{=} \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}). \tag{2}$$

Using this expression for δW, derive a necessary and sufficient condition for stability of the equilibrium. To simplify the analysis and save you time, I will tell you that the most easily destabilized displacement is of the form $\boldsymbol{\xi}(x)e^{iky}$ ($k \neq 0$), where $\boldsymbol{\xi}$ is a vector. (It should not be surprising that $k_z = 0$ for the most easily destabilized mode, because such a mode does not expend energy bending the magnetic field lines.) In your analysis, you may assume that the displacement takes this form.

Hint: After you have evaluated δW in Cartesian coordinates, use the equilibrium equation to eliminate ρ from the expression. This will lead to the cancellation of several terms.

Part II, Question 3: Gyrokinetics [20 points]

In the absence of finite-ion-Larmor-radius effects ($T_i = 0$), and for constant magnetic field $\boldsymbol{B} = B\widehat{\boldsymbol{z}}$, the electrostatic gyrokinetic equation for gyrocenters of species s is

$$\frac{\partial F_s}{\partial t} + v_\parallel \nabla_\parallel F_s + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} F_s + \left(\frac{q}{m}\right)_s E_\parallel \frac{\partial F_s}{\partial v_\parallel} = 0, \tag{1}$$

where $\boldsymbol{V}_E \stackrel{\text{def}}{=} (c/B)\widehat{\boldsymbol{z}} \times \boldsymbol{\nabla}\phi$ and $E_\parallel \stackrel{\text{def}}{=} -\nabla_\parallel \phi$, ϕ being the electrostatic potential. The associated gyrokinetic Poisson equation is approximately

$$-\varepsilon_\perp \nabla_\perp^2 \phi = 4\pi\rho^G, \tag{2}$$

where

$$\varepsilon_\perp \stackrel{\text{def}}{=} \omega_{pi}^2/\omega_{ci}^2 = \rho_s^2/\lambda_{De}^2 \quad \text{(perpendicular dielectric constant)}, \tag{3a}$$

$$\rho_s \stackrel{\text{def}}{=} c_s/\omega_{ci} \quad \text{(sound radius)}, \tag{3b}$$

$$c_s \stackrel{\text{def}}{=} (ZT_e/m_i)^{1/2} \quad \text{(sound speed)}, \tag{3c}$$

$$\lambda_{De} \stackrel{\text{def}}{=} (T_e/4\pi n_e e^2)^{1/2} \quad \text{(electron Debye length)}, \tag{3d}$$

and ρ^G is the gyrocenter charge density obtained by integrating F_s over velocity and appropriately summing over species.

(a) [4 points] By integrating Eq. (1) over velocity, show that the continuity equation for the ion gyrocenter density n_i^G is approximately

$$\partial_t n_i^G + \boldsymbol{\nabla} \cdot (\boldsymbol{V}_E\, n_i^G) \approx 0. \tag{4}$$

Briefly justify the physical approximation that was made in obtaining this result.

(Problem continues on next page.)

(b) [10 points] Assume that the ions have a weak background density gradient in the $-x$ direction such that the density scale length $L_n \stackrel{\text{def}}{=} -(d \ln n_i^G / dx)^{-1}$ can be considered to be constant. Use linear analysis of the ion gyrocenter continuity equation (4) and the gyrokinetic Poisson equation (2) to derive the standard drift wave dispersion relation

$$\omega_{\boldsymbol{k}} = \frac{\omega_*(\boldsymbol{k})}{1 + k_\perp^2 \rho_s^2}, \tag{5}$$

where

$$\omega_*(\boldsymbol{k}) \stackrel{\text{def}}{=} \frac{k_y \rho_s c_s}{L_n}. \tag{6}$$

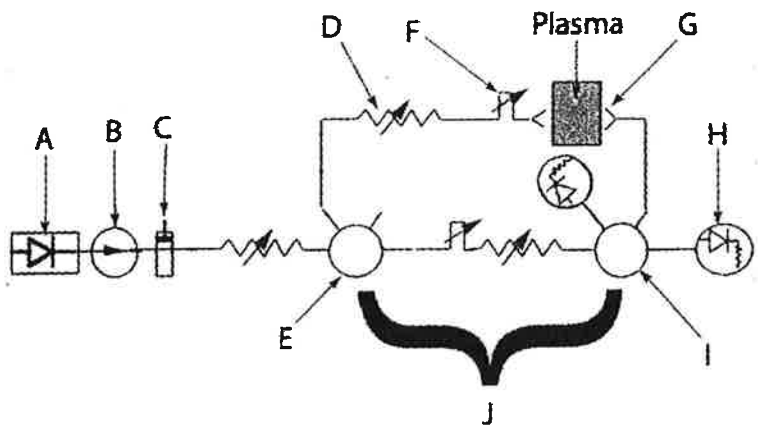
You may assume that

- the electrons obey adiabatic response ($\delta n_e / n_e \approx e\delta\phi / T_e$, where the δ's signify first-order quantities);
- $Z = 1$.

(c) [6 points] What is the physical significance of the $k_\perp^2 \rho_s^2$ correction to the dispersion relation? Be as complete as possible within the scope of a six-minute discussion. At the very least, give the name of the physical effect. If you have time, derive the term on the left-hand side of Eq. (2).

Part II, Question 4:
Experimental Methods
[55 points]

Microwave interferometry can be used to measure plasma density.



Consider the diagram above of a Mach–Zender-type interferometer operating at 35 GHz.

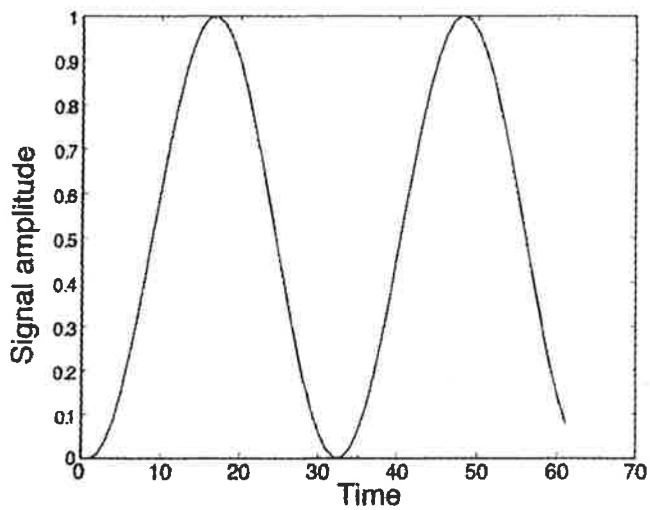
(a) **[15 points]** Explain the principle of interferometry as a means to measure the density of an unmagnetized plasma. *Equations are expected.* Above what density will the above system not work?

(b) **[5 points]** If the plasma were composed of a strongly electronegative gas, such as chlorine, that had the composition $n_e = n_{\mathrm{Cl}^-} = \frac{1}{2} n_{\mathrm{Cl}^+}$, how would your explanation in part (a) change?

(c) **[20 points]** Name each of the components A–J and explain their functions in the diagnostic.

(Problem continues on next page.)

(d) [15 points] Consider the measured signal shown in the figure below for a low-temperature helium plasma. Sketch onto the figure two possible interpretations of the density behavior and explain why the inferred $n_e(t)$ is not unique. (Assume that both n_e and the initial phase difference between the two arms are zero at $t = 0$.)



Part II, Question 5:
Kinetic Theory
[45 points]

Consider a homogeneous thermal plasma composed of mostly electrons and protons, but also containing some impurity ions of atomic charge A_I and mass Z_I. You are working with someone who has built a diagnostic (for either a real or simulated plasma) that measures the wave-number and frequency spectrum of the thermal fluctuations of the electrostatic potential Φ.

(a) [5 points] She claims that there is a low-frequency component of the fluctuations that depends on the density of the impurity ions. Briefly state why this could be true.

(b) [10 points] She also builds a diagnostic that measures the equal-time, two-point correlation function $\langle \Phi(\boldsymbol{x} + \boldsymbol{\rho}) \Phi(\boldsymbol{x}) \rangle$. Approximate the plasma as having a finite volume V with periodic boundary conditions. Show how this two-point correlation function can be related to the wave-number spectrum of the fluctuations.

(c) [15 points] She modifies the diagnostic so that it filters out the high-frequency components and can measure the (filtered) equal-time, two-point correlation function of the fluctuations due just to the impurity ions.

Calculate the wave-number spectrum of these impurity-produced fluctuations by assuming that the shielding cloud around each impurity ion is due to a simple Boltzmann response of the background electrons and protons, ignoring shielding by other impurity ions, and assuming that the impurity ions are statistically independent.

(d) [15 points] Calculate the mean-square amplitude of the electrostatic fluctuations due to these impurity ions. (You can take the continuous limit to turn sums into integrals.)