DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 19, 2008

9 a.m. - 1 p.m.

- Answer all problems.
  Problem 5 has a choice of (A) or (B). Answer only one.

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

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Problems for Part I, May 19, 2008

1)	Planar Diode	40 points
2)	High Harmonic Fast Waves in NSTX	40 points
3)	Neoclassical Theory	40 points
4)	Cylindrical Plasma	20 points
5A)	Applied Math	
	OR	
5B)	Computational Plasma Physics	40 points
		Total - 180 points

PART 1, QUESTION 1

1. [40 points]

A planar diode in which the cathode is a copious mitter of electrons is limited in the amount of current it carries because of space-charge effects. Assuming divergence-free steady-state flow in one dimension, one can derive the important "3/2" law for space-charge limited diodes, namely, that the maximum current is proportional to the voltage to the 3/2 power. This "3/2" law works for 1-D flow of a non-neutral stream of ions or electrons.

In this problem, suppose a small variation: Suppose instead that a source of ions situated at $x = 0$ emits ions in the x direction at speed v_0 rather than at rest. Suppose further that there is a voltage drop of ϕ_0 bewtween $x = 0$ and $x = L$. The ions are extracted at $x = L$, and the question is what is the maximum steady state ion current J that can be extracted as a function of v_0, ϕ_0, and L, as well as ion charge q and mass m.

(a) (5 pts) For a steady state flow of ions beginning with velocity v_0 at $x = 0$, sketch the potential as a function of x. [You can do this step now based on physical intuition, or you can solve exactly and come back to this after you do part (d).]

(b) (15 pts) Show that the potential must be of the form:

$$\left[\frac{d\phi(x)}{dx}\right]^2 = \alpha\left[\phi_M - \phi(x)\right]^{1/2} + \beta,$$

where α is a constant which is function of the current J, where ϕ_M is a constant which is a function of ϕ_0 and v_0, and where β is a constant of integration. Determine α and ϕ_M.

Hint: a simple identity you might find useful is:

$$\frac{d\phi(x)}{dx}\frac{d^2\phi(x)}{dx^2} = \frac{d}{dx}\left\{\frac{1}{2}\left[\frac{d\phi(x)}{dx}\right]^2\right\}.$$

(c) (5 pts) Argue from physical principles that the maximum steady state current will be extracted when β is set to zero.

(d) (15 pts) Derive an expression for the saturation current J. Show that, in the limit $v_0 \longrightarrow 0$, the saturation current obeys the familiar "3/2" scaling law.

Part I, Question 2

High Harmonic Fast Waves in NSTX – 40 points total

Recent experiments on NSTX have shown that the core heating efficiency of high harmonic fast waves (HHFW) depends on the launched parallel wave number, k_z , and the edge density. The density at which the launched waves can begin to propagate is proportional to $B \times k_{//}^2 \,/\, \omega^2$ (this comes from the $n_{//}^2$=R cutoff of the waves). Furthermore, for these waves, the energy flow is directed primarily parallel to the vessel wall rather than radially into the plasma. In essence, for a given density and equilibrium magnetic field profile, waves with a lower k_z begin to propagate closer to the antenna than those with higher k_z , thereby potentially losing more power in the edge due to interactions with the nearby vessel structures.

In this problem, by calculating the direction of the group velocity for the high harmonic fast waves in NSTX, you will show that the launched power flows mainly parallel to the equilibrium magnetic field, and hence the vessel itself, in the edge regions of the discharge. You may treat the edge region as a cold plasma slab, with the equilibrium magnetic field pointing in the Z-direction. You may also assume that $E_z \sim E_{//} \approx 0$ and that the plasma is composed only of deuterium and electrons. Other parameters of interest include the edge density (3×10^{12} cm^{-3}), the wave frequency (30 MHz), the parallel wave vector (k_z = 14 m^{-1}), and the edge magnetic field (3 kG).

1.) [5 points] In the cold plasma limit, write down or derive the dispersion relation for the high harmonic fast waves in terms of R, L, and S.

2.) [15 points] Derive an expression for $\dfrac{v_{g\perp}}{v_{g//}}$ in terms of $n_\perp$, $n_{//}$, and S.

3.) [15 points] Give approximate numerical values for ω/Ω_D, S, $n_{//}$, and $n_\perp$. Note, to estimate $n_\perp$, you may assume that $n_\perp >> n_{//}$ to simplify the dispersion relation you derived above.

4.) [5 points] Finally, give a numerical estimate of the approximate angle between the group velocity and the vessel wall.

Neoclassical Theory [40 points]

In an axisymmetric toroidal plasma, the conservation of canonical angular momentum can be used to estimate key neoclassical transport properties such as the inward particle pinch velocity ("Ware Pinch") and the "banana" excursion of trapped particles in the long-mean-free-path banana regime. The conservation of canonical angular momentum, P_ζ, can be expressed in terms of the poloidal flux function, ψ, as follows:

$$P_\zeta = mR\mathrm{v}_\zeta + \frac{e}{c}\psi$$

[10 pts.] (a) Show that $\partial\psi/\partial t = -\,\mathbf{v}\bullet\nabla\psi$ by assuming $\mathrm{v}_\zeta \approx \mathrm{v}_{||}$.

[10 pts.] (b) Estimate the trapped-particle radial velocity by using Faraday's Law together with the result from part (b).

[5 pts.] (c) Using $\mathbf{B} = \nabla\times\mathbf{A}$, now express P_ζ in terms of B_θ.

[5 pts.] (d) Expand around r_0, the mean radius of a trapped-particle orbit, to express the result from part (d) in terms of the trapped-particle radial excursion,

$$\Lambda \equiv r - r_0.$$

[10 pts.] (e) Taking $\mathrm{v}_\zeta \approx \mathrm{v}_{||} \approx \varepsilon^{1/2}\,\mathrm{v}$ (with $\varepsilon \equiv r/R_0$) and $B_\theta \approx B_p$ (poloidal magnetic field), obtain an estimate for Λ in terms of the gyroradius.

Part I, Question 4

MHD Quickie (20 points)

Consider a cylindrical model of a large aspect ratio toroidal plasma, with cylindrical coordinates (r,θ,z), and with periodic boundary conditions imposed on the magnetic field: $\mathbf{B}(r,\theta,z)|_{z=L} = \mathbf{B}(r,\theta,z)|_{z=0}$, $L = 2\pi R_0$, where we identify $\phi = R_0 z$. Suppose that there is a uniform field in the z direction, $B_z\hat{z}$, and a current density in the z direction, $j_{z0}(1-r^2)\hat{z}$, such that $q = 1.5$ at $r = 0$. (We have chosen our normalization here to make $r = 1$ at the plasma boundary.)

(a) (9 pts.) Give an expression for $q(r)$. What is q at $r = 1$ (the plasma boundary)? For field lines on the $q = 2$ surface, give an expression for the poloidal angle along the field line, $\theta(\phi;\theta_0)$, where $\theta = \theta_0$ at $\phi = 0$. Let r_s be the radius of the $q = 2$ surface. Relative to the motion of the field lines at $r = r_s$, do the field lines at $r = r_s + \delta r$ (δr small) rotate in the poloidal direction more rapidly or less rapidly, as a function of ϕ, than those at $r = r_s$. What about the field lines at $r = r_s - \delta r$?

(b) (8 pts.) Now add a small perturbing field, $\boldsymbol{\delta}\mathbf{B}$, with $\delta B_r = \varepsilon(r)\, B_z \sin(2\theta - \phi)$. Along an unperturbed field line trajectory on the $q = 2$ surface, what is $\delta B_r(\phi;\theta_0)$? Sketch the shape of the flux surfaces near $r = r_s$ in the presence of the perturbation. (You do not need to give a quantitative estimate of the magnitude of the effect, just a qualitative picture of what the flux surfaces look like.) Sketch the unperturbed surface with a dashed line.

(c) (3 pts.) On the same picture, sketch the perturbed and unperturbed flux surfaces at the $q = 1.75$ flux surface.

PART 1, QUESTION 5A

Generals Math 40 points

Consider

$$x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + xy = 0$$

a) Find two possible asymptotic behaviors at $x \to 0$
b) Find two possible asymptotic behaviors at $x \to +\infty$
c) Find a Fourier-Laplace integral representation for the solution
d) Find an integration contour if $y(0) = 1$.

PART 1, QUESTION 5B

Computational Methods (40 points)

Analysis of a Finite Difference Equation:

Consider the scalar one dimensional diffusion equation, where D is a constant, and U is the unknown function of time and one spatial dimension; $U(t,x)$,

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \tag{1}$$

Now consider the following finite difference approximation to Eq. (1):

$$U_j^{n+1} = U_j^n + \frac{\delta t}{(\delta x)^2}\left[\theta(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) + (1-\theta)(U_{j+1}^n - 2U_j^n + U_{j-1}^n)\right] \tag{2}$$

where $0<\theta<1$ is a parameter. Here, we use the (standard) notation that $U_j^n = U(t^n, x_j)$, where $t^n = n\ \delta t$, and $x_j = j\ \delta x$, with δt and δx being the timestep and zone size, respectively.

1. (15 points) Show that the finite difference equation (2) is consistent with the partial differential equation (1) in the limit as δt and $\delta x \rightarrow 0$ for all values of θ in the range $0 \leq \theta \leq 1$.

2. (15 points) Use Von Neuman Stability Analysis, or another method of your choosing, to calculate the range of values of δt for which the method is stable for different values of θ.

3. (5 points) What would be the special benefit of using the value $\theta = 1/2$ in Equation (2)?

4. (5 points) Indicate how you would solve equation (2) each timestep for the new time value U_j^{n+1}.

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GENERAL EXAMINATION, PART II

MAY 20, 2008

9 a.m. - 1 p.m.

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Problems for Part II, May 20, 2008

1)	Experimental Methods	40 points
2)	Experimental Plasma Physics	35 points
3)	General Plasma Phenomena	30 points
4)	Irreversible Processes	35 points
5)	MHD Stability	40 points
		Total - 180 points

Part II, Question 1

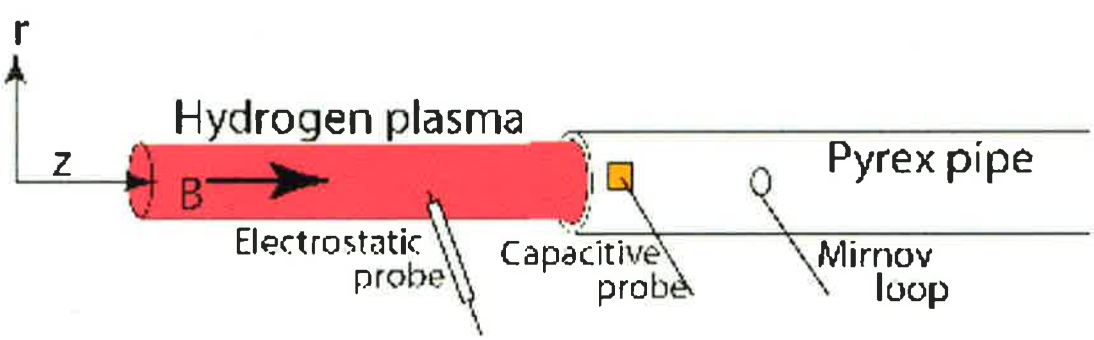
Experimental methods (40 points)

Consider a highly ionized, magnetized (B = 1 kG), pure hydrogen plasma column inside a thin-walled (1 mm) 10-cm-ID, 1-m-long Pyrex pipe. The plasma temperature and density are nearly constant with radius and axial position. (A plasma sheath exists at the Pyrex pipe wall.) The mean plasma parameters are: $T_e = 100$ eV, $T_i = 0.1$ eV, $n_e = 10^{13}$ cm^{-3}.

Scientists want to look at fluctuations at frequencies between the ion acoustic and LH ranges. The expected fluctuation levels in T_e, n_e and B are of order 0.1, 0.01 and 0.001 of their mean values.

The probes they have decided to use are:
1) an electrostatic probe of area 0.1 cm^2.
2) a capacitive probe (placed outside the Pyrex pipe) of area 1 cm^2
3) a single-turn Mirnov coil of area 0.5 cm^2, placed outside the Pyrex pipe

a. What frequency range, in units of Hz, will the experimentalists need to consider. (10 pts)
b. What parameters should the experimentalists measure with the **electrostatic** probe. What is (are) the expected signal size(s) due to fluctuations in each of T_e, n_e and B? (10 pts)
c. What causes the fluctuating signal measured with the **capacitive** probe? (Describe the physics.) What is the expected signal size due to fluctuations in each of T_e, n_e and B? (10 pts)
d. What parameter(s) could the experimentalists measure with the **Mirnov** probe? What is the expected signal size due to fluctuations in each of T_e, n_e and B? (10 pts)



Part II, Question 2
Experimental plasma physics
35 points

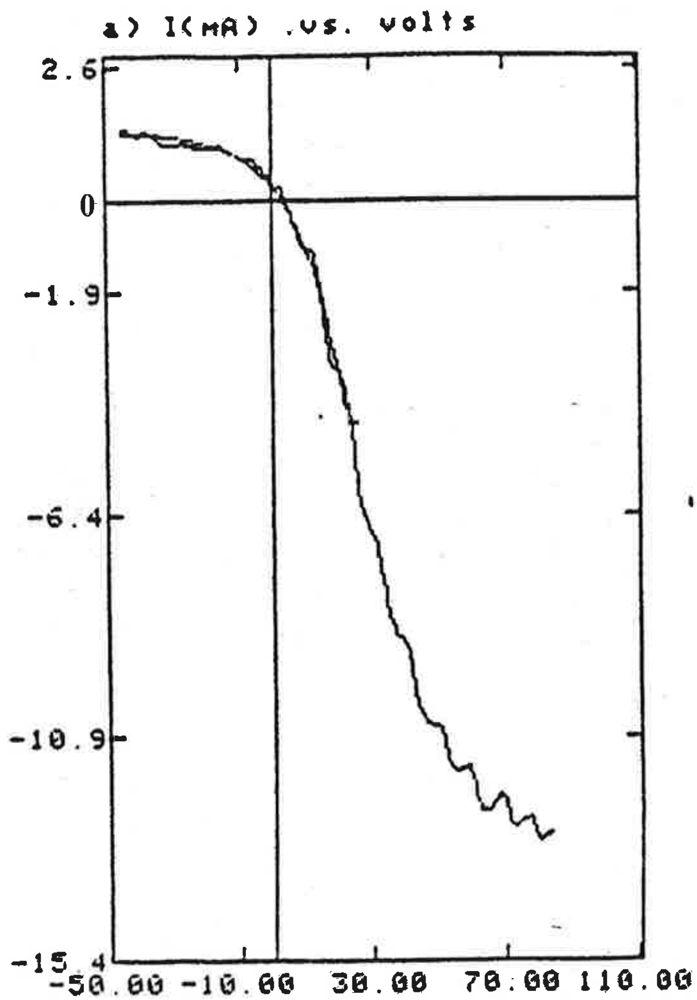
1. Using the Langmuir probe characteristic shown below:

a) Identify the ion and electron saturation current regions in the probe trace.

b) Indicate the approximate values for the floating potential and space potential.

c) If the ion species is argon, use the results of (b) to estimate the electron temperature.

d) If the probe tip (the conductor immersed in the plasma) is 1 mm in diameter and 5 mm long, what is the approximate value of the plasma density (again, assume singly charged argon for the ionic species)? Assume Te >> Ti. Are sheath expansion effects likely to affect the accuracy of the density estimate? Explain.



Part II, Question 2
Experimental plasma physics

(2)

In some plasma, one can consider doing interferometry with the extraordinary mode. It is of interest to calculate the error occurring in the deduced density if we use the expression for the ordinary mode index of refraction when really the mode used is the extraordinary mode. Consider perpendicular propagation and calculate an approximation for the difference in refractive index between the ordinary and extraordinary wave for $\omega >> \omega_p$. Calculate the fractional error in using $N^2_o = 1 - X$, where $(X = \omega^2_p/\omega^2)$, to determine density if the extraordinary mode is used in a plasma with $n = 10^{20}\ m^{-3}$, $B = 6$ T, and $f = 2x10^{12}$ Hz.

Other useful expressions: $N^2_x = 1 - \left[X(1-X)/(1-X-Y^2)\right]$, $Y = \Omega_c/\omega$, Ω_c is the cyclotron frequency.

$$\Delta\Phi = \omega/c \int (N-1)\, dl .$$

PART II, QUESTION 3

General Plasma Phenomena (30 points)

Consider the motion of a relativistic electron with charge $-e$, rest mass m, and kinematic energy $(\gamma-1)mc^2$ moving through a static, constant-amplitude, helical wiggler magnetic field specified by

$$\mathbf{B}_w^0(\mathbf{x}) = -B_w[\cos(k_w z)\hat{\mathbf{e}}_x + \sin(k_w z)\hat{\mathbf{e}}_y]. \tag{1}$$

Here, $B_w = const.$ and $\lambda_w = 2\pi/k_w = const.$ are the wiggler field amplitude and wavelength, respectively, $\gamma = (1+\mathbf{p}^2/m^2c^2)^{1/2}$ is the relativistic mass factor, and $\mathbf{p} = \gamma m \mathbf{v}$ is the kinematic momentum.

Denote the particle orbit that passes through the phase-space point $(\mathbf{x}, \mathbf{p})$ at time $t' = t$ by $(\mathbf{x}'(t'), \mathbf{p}'(t'))$, and assume that the transverse particle velocity is equal to zero in the absence of wiggler field, i.e., $v_x' = 0 = v_y'$ when $B_w = 0$.

(a)(15 points) Make use of the single-particle equation of motion to show that

$$\begin{aligned}
\gamma'(t') &= \gamma = const. \\
v_x'(t') &= a_w \frac{c}{\gamma}\cos[k_w z'(t')] \\
v_y'(t') &= a_w \frac{c}{\gamma}\sin[k_w z'(t')] \\
v_z'(t') &= v_z = const.
\end{aligned} \tag{2}$$

where $a_w = eB_w/mc^2k_w$ is the normalized wiggler amplitude, and $z'(t') = z + v_z(t'-t)$.

(b) (10 points) The spontaneous emission spectrum of the radiation emitted by the electron in the $+z$ diresction is given by

$$\frac{1}{T}\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c^3 T}\left|\int_0^T d\tau[v_x'(\tau)\hat{\mathbf{e}}_x + v_y'(\tau)\hat{\mathbf{e}}_y]\exp[ik_z z'(\tau) - i\omega\tau]\right|^2, \tag{3}$$

where ω is the emission frequency, k_z is the wavenumber, $\tau = t' - t$ is the shifted time variable, and $T = L/v_z$ is the length of time that the electron is in the interaction region (length $= L$). Here, $d^2I/d\omega d\Omega$ is the energy radiated per unit frequency interval per unit solid angle.

Make use of Eqs. (2) and (3) to calculate a closed expression for the spontaneous emission spectrum in the vicinity of the 'upshifted' emission frequency, $\omega - k_z v_z \approx +k_w v_z$.

(c) (5 points) Assume that the radiation emission corresponds to a light wave in vacuum ($\omega = ck_z$). Show that the spontaneous emission spectrum calculated in Part (b) is a maximum for radiation wavelength $\lambda_z = 2\pi/k_z$ given by

$$\lambda_z = \frac{(1+a_w^2)}{\gamma^2\beta_z(1+\beta_z)}\lambda_w, \tag{4}$$

where $\beta_z = v_z/c$, $\lambda_w = 2\pi/k_w$, and $\gamma = (1+\beta_z^2\gamma^2 + a_w^2)^{1/2}$. Therefore, for wiggler wavelength $\lambda_w = 2$ cm and amplitude $a_w = 0.3$, a 3 GeV beam of electrons ($\gamma \approx 8000$) is expected to produce radiation with wavelength $\lambda_z \approx 1.7\mathring{A}$.

PART II, QUESTION 4

2008 Plasma Physics General Exam

Irreversible Processes (35 minutes)

Consider a particle in a constant magnetic field, $\vec{B} = B_0\hat{z}$, subject to a drag force at the rate ν (taken to be a constant rate here), and to a rapidly fluctuating electric field in the x direction that we treat as white noise, $\langle E_x(t)E_x(t')\rangle = 2C_0\delta(t-t')$.

a. [5 points]. Write down a stochastic differential equation describing the evolution of the particle's velocity in the $\hat{x}$ and $\hat{y}$ directions (perpendicular to $\vec{B}$).

b. [10 points]. Write down the corresponding Fokker-Planck Equation for this problem.

c. [15 points]. From the Fokker-Planck equation, derive conservation laws for the time evolution of $\langle v_x^2\rangle$, $\langle v_y^2\rangle$, and $\langle v_x v_y\rangle$.

d. [5 points]. Calculate the average energy of a particle in statistical steady state.

MHD Long Problem (40 points)

In this problem, you will be asked to evaluate stability of ideal force-free equilibria (described by $\mathbf{j} = \mu(\mathbf{r})\mathbf{B}$) using energy principle and conservation of magnetic helicity.

(a) (15 minutes)

Prove conservation of magnetic helicity defined as $H = \int \mathbf{A} \cdot \mathbf{B} dV$, where $\mathbf{A}$ is vector potential of magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$. To be simple, V is a singly connected volume surrounded by a superconducting boundary with no interceptions of magnetic field.

(b) (10 minutes)

The energy integral of such plasmas due to displacement $\boldsymbol{\xi}(\mathbf{r})$ is given by

$$\delta W = \frac{1}{2}\int dV \left[\frac{\mathbf{Q}_\perp^2}{\mu_0} + \frac{\mathbf{B}^2}{\mu_0}(\nabla \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})^2 + \gamma p(\nabla \cdot \boldsymbol{\xi})^2 \right.$$
$$\left. -2(\boldsymbol{\xi}_\perp \cdot \nabla p)(\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}) - \mu(\mathbf{r})(\boldsymbol{\xi}_\perp^* \times \mathbf{B}) \cdot \mathbf{Q}_\perp \right]$$

where $\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$, p is plasma pressure, $\boldsymbol{\kappa}$ is magnetic curvature vector $\boldsymbol{\kappa} = \mathbf{b} \cdot \nabla \mathbf{b}$, $\mathbf{b} = \frac{\mathbf{B}}{|\mathbf{B}|}$. Among five terms in the energy integral, identify possible signs of each term.

(c) (15 minutes)

Derive a sufficient stability condition for the force-free equilibria using conservation of magnetic helicity.