DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 19, 2009

9 a.m. - 1 p.m.

- Answer all problems.
  Problem 3 has a choice of (A) or (B). Answer only one.

- The exam has been designed to require about 3 hours of work; however we have
  allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on
  questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question #
  and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate
  specifically how you would proceed if more time were available. If you do not attempt a
  particular problem, write on a sheet of paper "I have not attempted Problem _____ " and
  sign your name.

- All work on this examination must be independent. No assistance from other
  persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.
Problems for Part II, May 19, 2009

1) Elementary/General Plasma Physics 25 points

2) Kinetic Theory and Irreversible Processes 40 points

3A) Applied Mathematics

OR

3B) Computational Physics 40 points

4) MHD Physics (Quickie) 15 points

5) Experimental Methods 40 points

6) General Fusion Physics 20 points

Total - 180 points
Part II, Question 1

General Phenomena
(25 Points)

Consider the nonrelativistic motion of a particle with charge $q$ and mass $m$ in the uniform, time-dependent magnetic field $\mathbf{B} = B(t) \mathbf{e}_z$ inside a long, tightly-wound solenoid aligned in the $\mathbf{e}_z$ direction.

(a) (5 points) Show that the inductive electric field determined from $\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$ inside the solenoid is given by

$$\mathbf{E} = -\frac{1}{2c} \dot{\mathbf{B}}(t) (ye_x + xe_y),$$

where $xe_x + ye_y$ is the transverse displacement from the solenoid axis, and super-dot (\cdot) denotes $d/dt$.

(b) (5 points) Introduce the Larmor frequency $\Omega_L(t)$ and phase $\theta_L(t)$ defined by

$$\Omega_L(t) \equiv \frac{qB(t)}{2mc}, \quad \theta_L(t) \equiv \int_0^t dt' \Omega_L(t').$$

Show that the transverse equations of motion for $x(t)$ and $y(t)$ can be expressed as

$$\frac{d^2 x}{dt^2} + 2\frac{d\theta_L}{dt} \frac{dy}{dt} + \frac{d^2 \theta_L}{dt^2} y = 0, \quad (3)$$

$$\frac{d^2 y}{dt^2} - 2\frac{d\theta_L}{dt} \frac{dx}{dt} - \frac{d^2 \theta_L}{dt^2} x = 0. \quad (4)$$

(c) (10 points) The coupled equations of motion for $x(t)$ and $y(t)$ in the laboratory frame can be simplified by transforming Eqs. (3) and (4) to a frame of reference rotating with the Larmor frequency. Introduce the transverse orbits $X(t)$ and $Y(t)$ defined by

$$X + iy = (x + iy) \exp(-i\theta_L), \quad (5)$$

or equivalently, $x + iy = (X + iY) \exp(i\theta_L)$. Make use of Eqs. (3)-(5) to show that

$$\frac{d^2 X}{dt^2} + \Omega_L^2 X = 0, \quad (6)$$

$$\frac{d^2 Y}{dt^2} + \Omega_L^2 Y = 0. \quad (7)$$

(d) (5 points) It follows from Eqs. (6) and (7) that the transverse equations of motion for $X(t)$ and $Y(t)$ in the Larmor frame are decoupled, even in the presence of a time-dependent magnetic field $B(t)$. Indeed, the simplified forms of Eqs. (6) and (7) are readily amenable to direct analysis. For example, show that

$$X \frac{dY}{dt} - Y \frac{dX}{dt} = \text{const.} \quad (8)$$

is an exact constant of the motion, corresponding to conservation of canonical angular momentum in the Larmor frame.
2009 Plasma Physics General Exam

Irreversible Processes (40 minutes)

Consider electrostatic fluctuations in a statistically uniform plasma of periodic domain of volume \( V \).

a. [10 points]. Show how the 2-point correlation function for electrostatic potential fluctuations can be written in terms of the power spectrum, \( \langle |\Phi_k|^2 \rangle \), where

\[
\tilde{\Phi}(x) = \sum_k e^{i k \cdot x} \tilde{\Phi}_k
\]

b. [20 points]. The electric field spectrum in such a plasma in thermal equilibrium can be written as

\[
\frac{\langle |\tilde{E}_k|^2 \rangle}{8\pi} = \frac{1}{V} \frac{T}{2} \frac{1}{(1 + k^2 \lambda_T^2)}
\]

Using this, derive the rms normalized electric potential fluctuation amplitude, \( e\Phi_{rms}/T \), defined by

\[
\frac{e^2\Phi_{rms}^2}{T^2} = \frac{e^2}{T^2} \langle \Phi^2(x) \rangle
\]

Express \( e\Phi_{rms}/T \) in terms of common plasma parameters. (You should simplify your final answer by approximating \( k \)-summations with integrals.)

c. [10 points]. Briefly give a physical interpretation (or back-of-the-envelope estimate) of the result.
Math Problem - (40 points)

Consider the differential equation

\[
\frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 3y = 0.
\]

A. Find the leading asymptotic form of both real solutions for \( x \simeq 0 \).
B. Find the leading asymptotic form of both real solutions for \( x \simeq \infty \).
C. Find an integral representation for a solution with \( y(0) = 1 \) and tending to zero at \( x \to +\infty \).
Part II, Question 3B

Computational Methods (40 points)

**Analysis of a Finite Difference Equation:**
Consider the scalar one dimensional convection equation, where \(a\) is a constant, and \(U\) is the unknown function of time and one spatial dimension; \(U(t,x)\),

\[
\frac{\partial U}{\partial t} + a \frac{\partial U}{\partial x} = 0
\]  

(1)

Now consider the following finite difference approximation to Eq. (1):

\[
U_j^{n+1} = U_j^n - a \frac{\delta t}{\delta x} \left[ \theta(U_j^{n+1} - U_{j-1}^{n+1}) + (1 - \theta)(U_j^n - U_{j-1}^n) \right]
\]  

(2)

where \(0<\theta<1\) is a parameter. Here, we use the (standard) notation that \(U_j^n = U(t^n, x_j)\), where \(t^n = n \delta t\), and \(x_j = j \delta x\), with \(\delta t\) and \(\delta x\) being the time step and zone size, respectively.

1. (15 points) Show that the finite difference equation (2) is consistent with the partial differential equation (1) in the limit as \(\delta t\) and \(\delta x \to 0\) for all values of \(\theta\) in the range \(0 \leq \theta \leq 1\). What is the leading order truncation error?

2. (15 points) Use Von Neuman Stability Analysis to calculate the range of values of \(\delta t\) for which the method is stable for different values of \(a\):
   (a) for \(\theta=0\), and
   (b) for \(\theta=1\).

3. (10 points) Indicate how you would solve equation (2) for a value of \(\theta\) in the range: \(0 \leq \theta \leq 1\). ?
Part II, Question 4

MHD Quickie (15 points)

Combining the fluid equations for electrons and ions into a set of one-fluid equations, we get a momentum equation:

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{j} \times \mathbf{B} - \nabla p + \sigma \mathbf{E}. \]

a) Why is the \( \sigma \mathbf{E} \) term neglected in the MHD equations? Give an estimate for the relative magnitudes of the \( \sigma \mathbf{E} \) and \( \mathbf{j} \times \mathbf{B} \) terms under the assumptions made in MHD theory.

b) Does the equation \( \nabla \cdot (\epsilon_n \mathbf{E}) = \sigma \) remain valid in this regime? Is it used as one of the MHD equations? Explain.
Part II, Question 5

Long experimental plasma physics question
40 points total

I) (25 points) Consider an interferometer that operates at a radiation wavelength $\lambda_c = 10.6 \, \mu m$ (CO$_2$ laser) in the presence of spurious vibrations of the optical components. To compensate for these vibrations, interferometry is performed simultaneously using the same optical components at $\lambda_h = 0.633 \, \mu m$ (HeNe laser). The HeNe interferometer is affected much less than the CO$_2$ by the plasma phase shift, but still somewhat. If $\omega >> \omega_p$ for both wavelengths:

(a) Derive an expression for plasma density $\int n_e \cdot dl$ in terms of the phase shifts $\phi_c$ and $\phi_h$ of CO$_2$ and HeNe interferometers.

(b) If $\phi_h$ can be measured to an accuracy of $+/- \pi$, what uncertainty does this introduce into the plasma density measurement?

(c) Evaluate the fractional error in measuring a 1 m thick plasma of density $10^{20} \, m^{-3}$, assuming $\phi_c$ is measured exactly.

Hint:
$\phi_{\text{plasma}} = \frac{e^2 \lambda_c}{4\pi \varepsilon_0} \int n_e \cdot dl$

II) (15 points) You have been asked to design an experiment for a space shuttle mission. The experiment involves two shuttles in very close orbit connected by a long insulated cable. A differential voltage bias will be applied between the shuttles and the current collected will be monitored as a function of the applied bias.

The mission will be flown at 200 km altitude, near the lower boundary of the F layer, where the local electron density is expected to be between $10^5$ and $10^6 \, cm^{-3}$. Since the plasma in this region is primarily generated by photoionization, electrons of $1 - 2 \, eV$ are expected, and the ionic species is predominantly singly ionized cold oxygen. The earth’s magnetic field can be neglected (by declaration). The conducting (collection) area of each shuttle is 20 m$^2$.

(a) Calculate the voltage and current requirements for the power supply which provides the bias.

(b) Draw the current-voltage characteristic which you would expect if the electron density were $10^6 \, cm^{-3}$, and the electron temperature 2 eV.
Part II, Question 6

General Fusion Question: 20 minutes

The p - $^{11}$B (proton-boron) reaction is:

$$P + ^{11}B \rightarrow 3 \ ^{4}He + 8.7 \text{ MeV}$$

For fixed $\beta$ (ratio of plasma to magnetic field pressure), (magnetic field strength), and fixed $T_p = T_B = T_e$, but not fixed $n_e$, what are the ratios $n_p/n_e$ and $n_B/n_e$ that maximize the fusion power? You may ignore the presence of any impurities, including the He ash.