DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

MAY 19, 2009

9 a.m. - 1 p.m.

- Answer all problems.
  Problem 3 has a choice of (A) or (B). Answer only one.

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Problems for Part II, May 19, 2009

1)	Elementary/General Plasma Physics	25 points
2)	Kinetic Theory and Irreversible Processes	40 points
3A)	Applied Mathematics	
	OR	
3B)	Computational Physics	40 points
4)	MHD Physics (Quickie)	15 points
5)	Experimental Methods	40 points
6)	General Fusion Physics	20 points
		Total - 180 points

Part II, Question 1

General Phenomena
(25 Points)

Consider the nonrelativistic motion of a particle with charge q and mass m in the uniform, time-dependent magnetic field $\mathbf{B} = B(t)\,\mathbf{e}_z$ inside a long, tightly-wound solenoid aligned in the $\mathbf{e}_z$ direction.

(a) (5 points) Show that the inductive electric field determined from $\nabla \times \mathbf{E} = -c^{-1}\partial\mathbf{B}/\partial t$ inside the solenoid is given by

$$\mathbf{E} = -\frac{1}{2c}\dot{\mathbf{B}}(t)\left(-y\mathbf{e}_x + x\mathbf{e}_y\right), \tag{1}$$

where $x\mathbf{e}_x + y\mathbf{e}_y$ is the transverse displacement from the solenoid axis, and super-dot ($\dot{}$) denotes d/dt.

(b) (5 points) Introduce the Larmor frequency $\Omega_L(t)$ and phase $\theta_L(t)$ defined by

$$\Omega_L(t) \equiv -\frac{qB(t)}{2mc}, \quad \theta_L(t) \equiv \int_0^t dt'\,\Omega_L(t'). \tag{2}$$

Show that the transverse equations of motion for $x(t)$ and $y(t)$ can be expressed as

$$\frac{d^2x}{dt^2} + 2\frac{d\theta_L}{dt}\frac{dy}{dt} + \frac{d^2\theta_L}{dt^2}y = 0, \tag{3}$$

$$\frac{d^2y}{dt^2} - 2\frac{d\theta_L}{dt}\frac{dx}{dt} - \frac{d^2\theta_L}{dt^2}x = 0. \tag{4}$$

(c) (10 points) The coupled equations of moiton for $x(t)$ and $y(t)$ in the laboratory frame can be simplified by transforming Eqs. (3) and (4) to a frame of reference rotating with the Larmor frequency. Introduce the transverse orbits $X(t)$ and $Y(t)$ defined by

$$X + iY = (x + iy)\exp(-i\theta_L), \tag{5}$$

or equivalently, $x + iy = (X + iY)\exp(i\theta_L)$. Make use of Eqs. (3)-(5) to show that

$$\frac{d^2X}{dt^2} + \Omega_L^2 X = 0, \tag{6}$$

$$\frac{d^2Y}{dt^2} + \Omega_L^2 Y = 0. \tag{7}$$

(d) (5 points) It follows from Eqs. (6) and (7) that the transverse equations of motion for $X(t)$ and $Y(t)$ in the Larmor frame are decoupled, even in the presence of a time-dependent magnetic field $B(t)$. Indeed, the simplified forms of Eqs. (6) and (7) are readily amenable to direct analysis. For example, show that

$$X\frac{dY}{dt} - Y\frac{dX}{dt} = const. \tag{8}$$

is an exact constant of the motion, corresponding to conservation of canonical angular momentum in the Larmor frame.

Part II, Question 2

2009 Plasma Physics General Exam

Irreversible Processes (40 minutes)

Consider electrostatic fluctuations in a statistically uniform plasma of periodic domain of volume V.

a. [10 points]. Show how the 2-point correlation function for electrostatic potential fluctuations can be written in terms of the power spectrum, $\langle|\tilde{\Phi}_k|^2\rangle$, where

$$\tilde{\Phi}(x) = \sum_k e^{i\vec{k}\cdot\vec{x}}\tilde{\Phi}_k$$

b. [20 points]. The electric field spectrum in such a plasma in thermal equilibrium can be written as

$$\frac{\langle|\tilde{\vec{E}}_k|^2\rangle}{8\pi} = \frac{1}{V}\frac{T}{2}\frac{1}{(1+k^2\lambda_D^2)}$$

Using this, derive the rms normalized electric potential fluctuation amplitude, $e\Phi_{rms}/T$, defined by

$$\frac{e^2\Phi_{rms}^2}{T^2} = \frac{e^2}{T^2}\langle\Phi^2(x)\rangle$$

Express $e\Phi_{rms}/T$ in terms of common plasma parameters. (You should simplify your final answer by approximating k-summations with integrals.)

c. [10 points]. Briefly give a physical interpretation (or back-of-the-envelope estimate) of the result.

Part II, Question 3A

Math Problem - (40 points)

Consider the differential equation

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = 0.$$

A. Find the leading asymptotic form of both real solutions for $x \simeq 0$.

B. Find the leading asymptotic form of both real solutions for $x \simeq \infty$.

C. Find an integral representation for a solution with $y(0) = 1$ and tending to zero at $x \to +\infty$.

Part II, Question 3B

Computational Methods (40 points)

Analysis of a Finite Difference Equation:
Consider the scalar one dimensional convection equation, where a is a constant, and U is the unknown function of time and one spatial dimension; $U(t,x)$,

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0 \qquad (1)$$

Now consider the following finite difference approximation to Eq. (1):

$$U_j^{n+1} = U_j^n - a\frac{\delta t}{\delta x}\left[\theta(U_j^{n+1} - U_{j-1}^{n+1}) + (1-\theta)(U_j^n - U_{j-1}^n)\right] \qquad (2)$$

where $0<\theta<1$ is a parameter. Here, we use the (standard) notation that $U_j^n = U(t^n, x_j)$, where $t^n = n\,\delta t$, and $x_j = j\,\delta x$, with δt and δx being the time step and zone size, respectively.

1. (15 points) Show that the finite difference equation (2) is consistent with the partial differential equation (1) in the limit as δt and δx → 0 for all values of θ in the range $0 \le \theta \le 1$. What is the leading order truncation error?

2. (15 points) Use Von Neuman Stability Analysis to calculate the range of values of δt for which the method is stable for different values of a:
   (a) for $\theta=0$, and
   (b) for $\theta=1$.

3. (10 points) Indicate how you would solve equation (2) for a value of θ in the range: $0 \le \theta \le 1$. ?

Part II, Question 4

MHD Quickie (15 points)

Combining the fluid equations for electrons and ions into a set of one-fluid equations, we get a momentum equation:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\bullet\nabla\mathbf{v}\right)=\mathbf{j}\times\mathbf{B}-\nabla p+\sigma\mathbf{E}.$$

a) Why is the $\sigma\mathbf{E}$ term neglected in the MHD equations? Give an estimate for the relative magnitudes of the $\sigma\mathbf{E}$ and $\mathbf{j}\times\mathbf{B}$ terms under the assumptions made in MHD theory.

b) Does the equation $\nabla\bullet(\varepsilon_0\mathbf{E})=\sigma$ remain valid in this regime? Is it used as one of the MHD equations? Explain.

Part II, Question 5

Long experimental plasma physics question
40 points total

I) (25 points) Consider an interferometer that operates at a radiation wavelength $\lambda_c = 10.6\ \mu m$ (CO_2 laser) in the presence of spurious vibrations of the optical components. To compensate for these vibrations, interferometry is performed simultaneously using the same optical components at $\lambda_h = 0.633\ \mu m$ (HeNe laser). The HeNe interferometer is affected much less than the CO_2 by the plasma phase shift, but still somewhat. If $\omega >> \omega_p$ for both wavelengths:

(a) Derive an expression for plasma density $\int n_e * dl$ in terms of the phase shifts ϕ_c and ϕ_H of CO_2 and HeNe interferometers.

(b) If ϕ_H can be measured to an accuracy of +/- π, what uncertainty does this introduce into the plasma density measurement?

(c) Evaluate the fractional error in measuring a 1 m thick plasma of density 10^{20} m^{-3}, assuming ϕ_c is measured exactly.

Hint:

$$\phi^{plasma} = \frac{e^2 \lambda}{(4\pi c^2 m_e \varepsilon_0)} \int n_e * dl$$

II) (15 points) You have been asked to design an experiment for a space shuttle mission. The experiment involves two shuttles in very close orbit connected by a long insulated cable. A differential voltage bias will be applied between the shuttles and the current collected will be monitored as a function of the applied bias.

The mission will be flown at 200 km altitude, near the lower boundary of the F layer, where the local electron density is expected to be between 10^5 and 10^6 cm^{-3}. Since the plasma in this region is primarily generated by photoionization, electrons of 1 – 2 eV are expected, and the ionic species is predominantly singly ionized cold oxygen. The earth's magnetic field can be neglected (by declaration). The conducting (collection) area of each shuttle is 20 m^2.

(a) Calculate the voltage and current requirements for the power supply which provides the bias.
(b) Draw the current-voltage characteristic which you would expect if the electron density were 10^6 cm^{-3}, and the electron temperature 2 eV.

Part II, Question 6

General Fusion Question: 20 minutes

The p – ^{11}B (proton-boron) reaction is:

$P + {}^{11}B \rightarrow 3\,{}^{4}He + 8.7$ MeV

For fixed β (ratio of plasma to magnetic field pressure), (magnetic field strength), and fixed $T_p = T_B = T_e$, but not fixed n_e, what are the ratios n_p/n_e and n_B/n_e that maximize the fusion power? You may ignore the presence of any impurities, including the He ash.