

**DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS**

GENERAL EXAMINATION – PART I

MAY 9, 2011

9 A.M. – 1 P.M.

Answer all problems.

Problem 1 has a choice of (A) or (B). Answer only one.

Problem 5 has a choice of (A) or (B). Answer only one.

Part I has been designed to require about 3 hours to complete. However, an extra hour has been allowed. Thus the total time allotted for Part I is 4 hours. Scores assigned to the questions are intended to be proportional to the time required to obtain the solution.

Begin each numbered problem in a new test booklet. Put your name and the Question # on every booklet title page.

If you do not have adequate time to answer questions in a form that is acceptable to you, please describe how you would proceed to solve the problem if more time were available.

If you have not attempted a particular problem, please write on a sheet of paper “I have not attempted Problem #_____” and sign your name.

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Problems for Part I

(1A) Introductory Plasma Physics (Quickie)

OR

15 Points

(1B) Applied Mathematics (Quickie)

(2) Waves and Instabilities

45 Points

(3) Plasma Diagnostics

20 Points

(4) Neoclassical Physics

60 Points

(5A) Experimental Plasma Physics

OR

40 Points

(5B) Applied Mathematics

Total – 180 Points

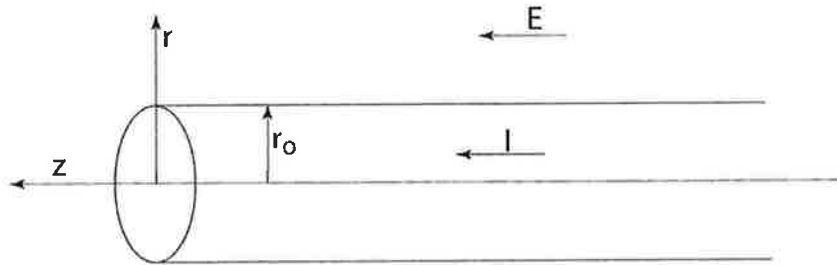
Part 1 – Question 1A

Introductory Plasma Physics (Quickie) Drift Motion [15 pts]

A current, I , flows in the $+z$ direction. It has a uniform radial distribution inside $r = r_0$ and is zero for $r > r_0$. (See the figure below.) Additionally, there is a uniform electric field (in all of space), also pointing in the $+z$ direction.

Consider two low energy ions, one initially beyond r_0 and the other inside r_0 . (Low energy means that the gyroradius, ρ_i , of an ion at $r = r_0$ is $\rho_i(r_0) \ll r_0$.) Assume that both ions, each of charge q , have zero azimuthal velocity and are initially *not* within a gyroradius of r_0 .

- (a) Graph the amplitude of the magnetic field generated by I as a function of radius r for $0 < r < 5r_0$ and the magnetic-field gradient as function of r . (3 pts)
- (b) Sketch each ion's motion for $E = 0$. What drift causes this motion? (3 pts)
- (c) Sketch each ion's motion for $E > 0$ but $E \ll v_{i0} B(r_0)$ where v_{i0} is the ion's initial velocity and $B(r_0)$ is the magnetic field at r_0 . (4 pts)
- (d) Does either ion ever reach $r = 0$? If yes, sketch the subsequent motion. (Hint: Consider a particle initially at rest at $r = 0$ then one with a small initial velocity in the r direction.) (5 pts)



Part 1 – Question 1B

Applied Mathematics (Quickie)

[15 Points]

Consider

$$\frac{d^2\psi}{dx^2} + Q(x)\psi = 0$$

Assume an exponential form for the solution $\psi = e^S$ and find S and a condition on Q such that you obtain asymptotic solutions. Find S to second order. At what values of Q do the solutions clearly fail?

Part 1 – Question 2

Waves and Instabilities

Langmuir Waves [45 points]

In this problem you are asked to rederive the dispersion relation $\omega(k)$ for Langmuir waves without using the Vlasov equation, assuming there is no dissipation. For simplicity, assume also that the electron motion is one-dimensional and neglect the ion motion.

- (a) [5 points] Identify the reference frame K' , where the wave potential φ is static. Using energy conservation for the electron motion in that frame, express the local particle velocity $w(\varphi, w_0)$ in frame K' as a function of the local wave potential $\varphi(x)$ and of the velocity w_0 outside the field (where $\varphi = 0$).
- (b) [25 points] The spatial density dn of electrons from a given interval $(w, w + dw)$ is inversely proportional to $|w|$, i.e., $|w| dn = |w_0| n_0 f_w(w_0) dw_0$, where n_0 is the unperturbed total density, and $f_w(w_0)$ is the corresponding velocity distribution. Using $w(\varphi, w_0)$ from the solution in (a), show that the linear dispersion relation can then be expressed as

$$1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{+\infty} \frac{f_0(v)}{(v - \omega/k)^2} dv = 0, \quad (1)$$

where $f_0(v)$ is the distribution of velocities $v = w_0 + \omega/k$, and resonant particles are neglected. *Hint:* Calculate the total charge density, substitute it into Poisson's equation, and linearize the latter with respect to φ , neglecting particles with very small w_0 .

- (c) [15 points] Simplify the above dispersion relation by expanding the integrand in kv/ω and keeping only the leading-order thermal corrections. Show how the familiar dispersion relation is obtained for (dissipationless) Langmuir waves.

Part 1 – Question 3

Plasma Diagnostics (Quickie) [20 Points]

- (a) Derive the Thomson scattering cross-section.
- (b) Calculate the fraction of photons incoherently scattered from a 1 cm path length of laser beam from a plasma with an electron density of $2 \times 10^{20} \text{ m}^{-3}$ with a solid angle of detection of 0.01 sr.

Useful constant: Classical electron radius = $2.8 \times 10^{-15} \text{ m}$.

Part 1 – Question 4

Neoclassical Transport and Drift Waves [60 Points]

This problem has two parts: **Part 1** deals with a magnetically-confined axisymmetric toroidal plasma with the driving forces being a scalar pressure gradient in the radial direction (assume no temperature gradient) and a parallel electric field, E_{\parallel} . **Part 2** deals with basic drift waves in a simple slab geometry.

Part 1: Estimate the terms in the “neoclassical” two-by-two matrix that relates the fluxes -- trapped-particle flux and charge flux (i.e., current density) to the driving forces (density gradient and parallel electric field) in the “banana regime.”

[25 pts.] (a) Give the trapped-particle flux using simplified (heuristic) estimates of the coefficients for the driving forces.

[15 pts.] (b) Give a simplified (heuristic) estimate for the neoclassical current density by specifying the coefficients for the driving forces.

Part 2: Electrostatic drift waves (and associated instabilities) are often invoked when neoclassical theory prove inadequate to account for the higher levels of transport often observed in toroidal experiments.

[15 pts.] (a) Estimate the perturbed density responses for the kinetic “adiabatic” electrons and the “cold fluid” ions respectively. Combine to give the simple drift-wave dispersion relation.

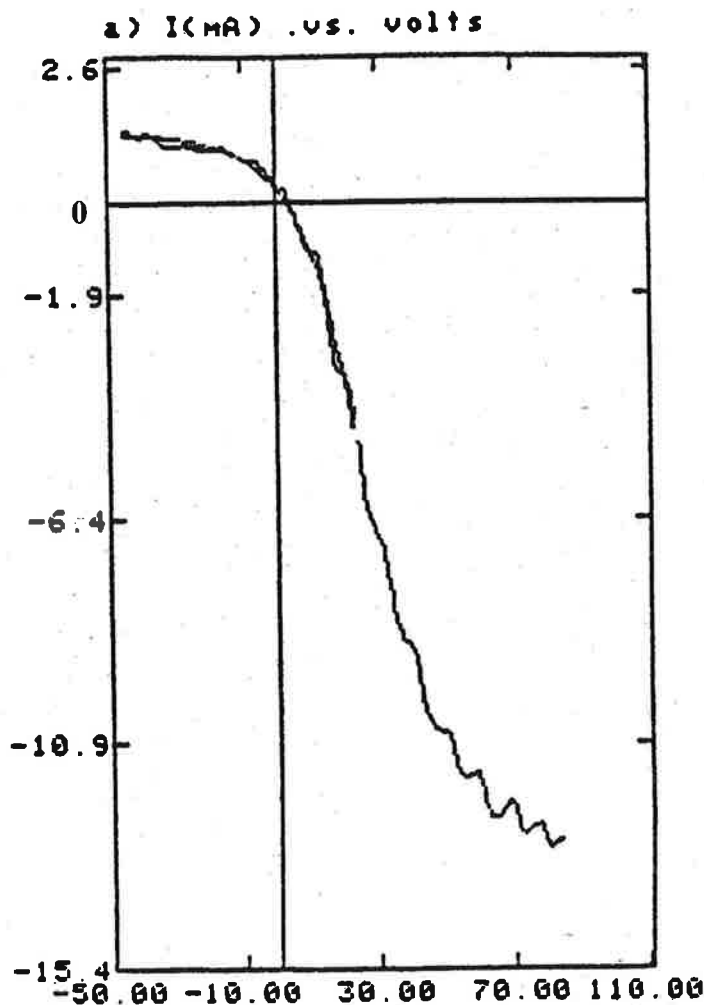
[5 pts.] (b) Justify the use of the “quasineutrality condition” relating the perturbed ion and electron density responses in part (a).

Part 1 – Question 5A

Experimental Plasma Physics [40 Points]

Using the single-tip Langmuir probe characteristic shown below:

- (a) Identify the ion and electron saturation current regions in the probe trace.
- (b) Give approximate values for the floating potential and space potential.
- (c) The ion species is argon (atomic mass 40), singly charged. Use the Langmuir trace to estimate the electron temperature.
- (d) If the probe tip (the conductor immersed in the plasma) is 1 mm in diameter and 5 mm long, what is the approximate value of the plasma density (reminder: singly charged argon)? Assume $T_e \gg T_i$. Are sheath expansion effects likely to affect the accuracy of the density estimate? Do you see any evidence for sheath effects in the probe trace? Explain.



Part 1 – Question 5B

Applied Mathematics

[40 points]

Consider

$$\frac{d^2y}{dx^2} - x \frac{dy}{dx} - y = 0$$

- (a) Find two possible asymptotic behaviors at $x \rightarrow +\infty$. (10 pts)
- (b) Find two possible asymptotic behaviors at $x \rightarrow 0$. (10 pts)
- (c) Find a solution $y(x)$ with $y \rightarrow 0$ for $x \rightarrow +\infty$ and give normalized asymptotic expressions for $x \rightarrow +\infty$ and $x \rightarrow 0$. (20 pts)

**DEPARTMENT OF ASTROPHYSICAL SCIENCES
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GENERAL EXAMINATION – PART II

MAY 10, 2011

9 A.M. – 1 P.M.

Answer all problems.

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Problem 3 has a choice of (A) or (B). Answer only one.

Part II has been designed to require about 3 hours to complete. However, an extra hour has been allowed. Thus the total time allotted for Part I is 4 hours. Scores assigned to the questions are intended to be proportional to the time required to obtain the solution.

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Problems for Part II

(1A) Kinetic Effects (Quickie)

OR

20 Points

(1B) MHD Physics (Quickie)

(2) Plasma Waves

30 Points

(3A) General Plasma Physics

OR

40 Points

(3B) Computational Physics

(4) MHD Physics

45 Points

(5) Irreversible Processes

45 Points

Total – 180 Points

Part 2 –Question 1A

Kinetic Effects (Quickie) [20 Points]

Most of the plasma ions incident on a solid surface will eventually come back as cold neutrals, but a fraction of them may be promptly reflected. Here we consider a simple 1-D model of this prompt reflection process. Consider the distribution function of ions $f(x, v, t)$ for $x > 0$, bounded by a wall at $x = 0$.

(a) Write the flux of incident ions into the wall as an integral involving the distribution function.

(b) Write down boundary conditions that model these reflections, assuming that the probability of an incident ion being reflected is α (independent of energy) and that the reflected energy of an ion is a fraction β of its incident energy.

Part 2 –Question 1B

Magnetohydrodynamics (Quickie) [20 Points]

In this problem, you are asked to derive the MHD equation of motion and Ohm's law, and to estimate the conditions under which they are valid.

- (a) [5 Points] Write down the equations of motion for electron and ion fluids, assuming that their pressures are scalar, and that the ions are singly charged.
- (b) [5 Points] Ignoring the electron mass, derive the one-fluid MHD equation of motion and the generalized Ohm's law from the force-balance equations in Part (a).
- (c) [5 Points] What are the two conditions under which the generalized Ohm's law obtained in Part (b) can be reduced to the regular form of Ohm's law used in MHD.
- (d) [5 Points] Are these conditions typically satisfied in fusion plasmas? Can these two conditions be combined into a single condition?

Part 2 –Question 2

General Phenomena

Plasma Waves [30 Points]

Consider an extraordinary electromagnetic wave propagating perpendicular to a uniform magnetic field $B_0 \mathbf{e}_z$. The plasma is cold and homogenous.

- (a) [20 Points] Show that the dispersion relation has a resonance near the upper hybrid frequency.
- (b) [10 Points] Prove that the wave polarization is approximately electrostatic near the upper-hybrid resonance frequency.

Part 2 –Question 3A

General Plasma Physics [40 Points]

This problem will consider the collisionless damping of a nearly cold plasma by resonant electrons and the self-consistent reaction of the wave on the electron velocity distribution function.

- (a) [12 points] Beginning with the 1D fluid equations for a cold infinite homogeneous plasma with stationary ions, derive the dispersion relation for small plasma oscillations at the plasma frequency $\omega_p^2 = q^2 n_0 / \epsilon_0 m$. You may assume that the oscillations are small enough that the electron density may be written as $n(x, t) = n_0 + \tilde{n}(x, t)$, where $\tilde{n}(x, t)$ may be treated as a small perturbation. Similarly, the electron fluid velocity may be assumed to be a small perturbation.
- (b) [8 points] For what initial conditions of the perturbed electron density, $\tilde{n}(x, t = 0)$, and perturbed electron velocity, $\tilde{v}(x, t = 0)$, will the excited electric field evolve as $E(x, t) = E_0 \cos(kx - \omega_p t)$?
- (c) [5 points] Suppose now that the electrons are not perfectly cold, but are nearly cold with a velocity distribution $f(v, t)$. Suppose that the tail of the electron distribution function interacts with the wave such that it causes the wave to damp through Landau damping. Consider the case in which as $t \rightarrow \infty$, the wave is not completely extinguished, but rather $E(x, t \rightarrow \infty)$ remains finite. Sketch for this case $f(v, t \rightarrow \infty)$.
- (d) [15 points] Show that an approximate condition for $E(x, t \rightarrow \infty)$ remaining finite despite Landau damping is:

$$v_{tr} > c \frac{\omega_p^3}{k^3} \left| \frac{\partial f_0}{\partial v} \right|_{v=\omega/k},$$

where $v_{tr} \equiv \sqrt{qE_0/mk}$ and $f_0(v) \equiv f(v, t = 0)$ and c is a constant of order 1. Explain briefly any assumptions made in arriving at this result.

Part 2 –Question 3B

Computational Physics:

Finite Difference Equation with Non-constant Thermal Conductivity [40 Points]

Consider the non-linear diffusion equation for the temperature T in one-dimensional slab geometry:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[\chi \frac{\partial T}{\partial x} \right], \quad (1.0)$$

where the thermal conductivity is proportional to the temperature gradient raised to the power p ,

$$\chi = \chi_0 \left(\frac{\partial T}{\partial x} \right)^p. \quad (1.1)$$

Here, χ_0 is a constant. Perform a linear analysis of the stability of the backward-time centered-space (BTCS) finite difference method where the thermal conductivity term is evaluated at the old time level, i.e.,

$$T^{n+1} = T^n + \delta t \frac{\partial}{\partial x} \left[\chi^n \frac{\partial T^{n+1}}{\partial x} \right]. \quad (1.2)$$

In Eq. (1.2), the superscript n is the time index, $t^n = n\delta t$, and centered spatial differences are to be used. Assume the temperature is slightly perturbed from its equilibrium value, $T^n = T^0 + \tilde{T}^n$, where T^0 satisfies the steady-state equation

$$\frac{\partial}{\partial x} \left[\chi_0 \left(\frac{\partial T^0}{\partial x} \right)^p \frac{\partial T^0}{\partial x} \right] = 0,$$

and $\tilde{T}^n \ll T^0$. What are the conditions for stability in the limit that the perturbation wavelength is small compared to the background gradients?

Part 2 –Question 4

Magnetohydrodynamics [45 Points]

(a) (5 points) A plasma cannot be confined by a purely toroidal axisymmetric field. Explain why this is the case in terms of the particle drift trajectories. The field is $\mathbf{B} = B_0 \nabla \phi$ in cylindrical coordinates (R, ϕ, z) , where R is the major radius ($R = 0$ is the symmetry axis), ϕ is the toroidal angle, and z is the coordinate in the vertical direction. To simplify things, take B_0 to be a constant (independent of spatial coordinates).

(b) (15 points) Now let's look at this in terms of the MHD equations. Using the same field as in part (a), solve for \mathbf{j}_\perp from the MHD equilibrium equation (with $\mathbf{v} = 0$), where \mathbf{j}_\perp is the component of \mathbf{j} perpendicular to the magnetic field. Assume that β is sufficiently small that we can neglect the effect of the pressure driven current on B_0 . Give an explicit expression for $\nabla \cdot \mathbf{j}$ in terms of B_0 and the derivative of the pressure. How does the expression relate to part (a) of this question?

(c) (10 points) Now let's add a toroidal current. How does this help us? Explain this both in terms of the drift trajectory picture and in terms of the effect on the MHD equations.

(d) (15 points) Write $\mathbf{j}_\parallel = \lambda \mathbf{B}$, where λ is a function of position to be determined. Assuming the same toroidal field as in (a), with finite poloidal field, what is the equation for λ that follows from the requirement that the current density must have zero divergence? Give an explicit solution of this equation in the large aspect ratio limit for an equilibrium with circular flux surfaces, $p = p(r)$, where r is the minor radius. Express the solution in terms of B_0 , dp/dr and the strength of the poloidal field B_p . What determines the constant of integration in the solution?

Irreversible Processes [45 points]

In magnetic confinement, one usually assumes good flux surfaces. But microturbulence can produce small, random magnetic fluctuations that can sometimes break the surfaces. If that happens, magnetic field lines can wander stochastically in the volume, and particles can get rapidly from the inside to the outside by streaming along the stochastic field lines.

Consider a simple slab model with a z -directed, constant magnetic field $B_z \hat{z}$. Let x be a direction of inhomogeneity, like a radial coordinate in a tokamak. In the absence of any additional magnetic fields, the flux surfaces are sheets in the y - z plane.

(The model we are building is not entirely realistic; it does not include magnetic shear. That does not matter for the purposes of this problem.)

Now add a small random magnetic perturbation $\delta \tilde{B}_x(z, t)$. The tilde denotes a random variable. (The z dependence would arise from the microscale structure of some underlying turbulence. Assume that $\delta \tilde{B}_x$ is statistically stationary in time.) The field line now wanders randomly in the x direction according to

$$\frac{\partial \tilde{x}}{\partial z} = \frac{\delta \tilde{B}_x(z, t)}{B_z} \equiv \delta b(z, t). \quad (1)$$

Furthermore, a test particle moves along the field line according to

$$\frac{d\tilde{\ell}}{dt} = \tilde{v}(t), \quad (2)$$

where ℓ is the distance along the field line and $\tilde{v}(t)$ is random because the particle can experience collisions. Because the field line is almost in the z direction (the magnetic perturbation is small), it will be adequate to approximate the parallel direction by the z direction and to write

$$\frac{d\tilde{z}}{dt} = \tilde{v}(t). \quad (3)$$

You may assume the standard (1D) Langevin model for the effects of collisions on the parallel motion of the test particle.

If one combines Eqs. (1) and (3) by using the chain rule, one finds an equation for the x motion of the test particle:

$$\frac{d\tilde{x}}{dt} = \tilde{v}(t) \delta b(\tilde{z}(t), t) \equiv \delta \tilde{V}(t). \quad (4)$$

We will use this equation to explore the stochastic motion of the test particle in the x direction.