DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

MAY 14, 2012

9 a.m. - 1 p.m.

- Answer all problems.
  Problem 1 has a choice of (A) or (B). Answer only one.

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, and question # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate <u>specifically</u> how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

Problems for Part I, May 14, 2012

1A)	General Plasma Physics (Quickie)	
	OR	10 points
1B)	Applied Mathematics (Quickie)	
2)	Waves and Instabilities	50 points
3)	Magnetic mirror	40 points
4)	Landau Damping	40 points
5)	Neoclassical Physics	40 points

Total - 180 points

Part I, Question 1A

Quickie: General plasma physics (10)

A single Langmuir probe is made of tungsten and has a surface area 1 cm^2. It is immersed in a large plasma, bounded by remote tungsten walls. The plasma consists of equal numbers of positrons and electrons. Each species has a central density of 10^8 cm^{-3} and temperature 1 eV. There are no neutral particles in the plasma.

1) Sketch the Langmuir probe characteristic, giving absolute values for the scales on the abscissa (I) and ordinate (V). (4)
2) What is the plasma potential with respect to the walls? (2)
3) What is the power density on the probe for biases of +50 and –50 V with respect to the wall? (4)

Part I, Question 1B

Applied Mathematics (Quickie) **10 Points**

Consider

$$\frac{d^2y}{dz^2} + Q(z)y = 0$$

Define an Anti-Stokes line and explain why it is important. How are these lines distributed near a first order zero of Q? Second order? First order singularity?

Part I, Question 2

Waves and Instabilities

Electrostatic waves in relativistic plasma [50 points]

Assuming real ω and k, calculate the longitudinal dielectric function $\epsilon(\omega, k)$ of ultrarelativistic electron plasma with the unperturbed electron distribution given by

$$f_0(\mathbf{p}) = f_0(p) = \frac{n_0 c^3}{8\pi T^3} \exp\left(-\frac{cp}{T}\right), \qquad \int_0^\infty 4\pi p^2 f_0(p) dp = n_0.$$

Hints: (i) Note that $f_0(p)$ is a three-dimensional momentum distribution, with $p = |\mathbf{p}|$. Choose $\mathbf{k}$ along z and use $p_\parallel = p\cos\theta$ for the momentum component along $\mathbf{k}$; here θ is the polar angle in the associated spherical coordinates. (ii) As usual, when taking integrals, you may find it convenient to employ the identity $x/(x-1) = 1 + 1/(x-1)$.

(a) [20 points] Write down the Vlasov equation for $f(\mathbf{x}, \mathbf{p})$. Use it to show that

$$\epsilon(\omega, k) = 1 + \frac{4\pi e^2}{k^2 T} \int_0^\infty f_0(p)\, 2\pi p^2\, dp \int_0^\pi \frac{\cos\theta}{\cos\theta - u} \sin\theta\, d\theta, \qquad u \equiv \frac{\omega}{kc}.$$

(b) [15 points] How should one understand the above (formally divergent) integral over θ? Find both $\epsilon' \equiv \mathrm{Re}\,\epsilon$ and for $\epsilon'' \equiv \mathrm{Im}\,\epsilon$ explicitly, by integrating over p and over $s \equiv \cos\theta$. Show that ϵ' exhibits a logarithmic singularity at $u = 1$.

(c) [15 points] Sketch both ϵ' and ϵ'' as functions of u. For this, you may want to derive asymptotics of $\epsilon(\omega, k)$ at $u \ll 1$ and at $u \gg 1$ first. Alternatively, you should be able to figure out what the result is, roughly, based on your knowledge of nonrelativistic plasmas. In either case explain your results qualitatively.

Generals Question 2012
May 2, 2012

1. [40 pts] Consider ions in a magnetic mirror machine with mirror axis in the $\hat{z}$ direction. Suppose a mirror ratio R. Furthermore, suppose that the magnetic field near the axis can be approximated as

$$\mathbf{B} = \begin{cases} B_0(1 + z^n/L^n)\hat{z}, & \text{if } z^n < c^n L^n \\ B_0(1 + c^n)\hat{z}, & \text{if } z^n > c^n L^n \end{cases}.$$

where we expressed the mirror ratio as $R = 1 + c^n$, where c is a constant an n is an even integer. Let $W_{\perp 0}$ and $W_{\| 0}$ be the perpendicular and parallel energies as the midplane ($z = 0$) is crossed.

(a) [8 pts] Derive the trapping condition in midplane ($z = 0$) energy coordinates.

(b) [3 pts] Sketch the trapping condition in the $W_{\perp 0}$-$W_{\| 0}$ plane.

(c) [5 pts] Show that the turning points for trapped ions obey: $z_T^2/L^2 = W_{\| 0}/W_{\perp 0}$.

(d) [4 pts] Suppose now that the magnetic field is changed slowly such that simultaneously $L \to \alpha L$ and $B_0 \to \beta B_0$. Calculate the new perpendicular midplane energy $W'_{\perp 0}$ in terms of α, β, $W_{\perp 0}$ and $W_{\| 0}$.

(e) [8 pts] Suppose again that the magnetic field is changed slowly such that simultaneously $L \to \alpha L$ and $B_0 \to \beta B_0$. Calculate the new parallel midplane energy $W'_{\| 0}$ in terms of α, β, $W_{\perp 0}$ and $W_{\| 0}$.

(f) [6 pts] For what initial midplane coordinates will particles initially trapped become untrapped as a result of this slow change in the magnetic field. Write your answer in terms of α, β, $W_{\perp 0}$, $W_{\| 0}$ and R.

(g) [2 pts] Sketch this condition in the $W_{\perp 0}$-$W_{\| 0}$ plane.

(h) [4 pts] Show that there are at least some initially trapped particles that become untrapped for $\beta < \alpha^{1/n}$.

Landau Damping (40 Points)

Consider the electrostatic perturbation of a homogeneous, unmagnetized plasma. Assume ions are motionless. For simplicity, we also assume that k is in the $x-$direction and $k>0$. The dispersion relation of the perturbation is

$$1+\frac{\omega_{pe}^2}{k^2}\int_L \frac{1}{\omega/k-u}\frac{\partial g_{e0}}{\partial u}\,du=0,$$

Where $g_{e0}\equiv\frac{1}{n_{e0}}\int dv_y dv_z f_{e0}$ is the unperturbed electron distribution function integrated over v_y and v_z.

a) [7 pts] The subscript "L" in the integral $\int_L du$ indicates that the integral is carried out along the Landau contour. What is the Landau contour?

b) [18 pts] From the dispersion relation show that if g_{e0} is a monotone-decreasing function of u^2, *i.e.*, $u\partial g_{e0}/\partial u<0$, then the system is stable with respect to the perturbation.

Neoclassical Physics 40 Points

This problem deals with the "neoclassical" properties of a plasma for a simplified model axisymmetric toroidal plasma with the driving forces being a pressure gradient in the radial direction (assume no temperature gradient) and a parallel electric field, $E_{||}$.

[10 pts.] (a) Give the dimensionless measure of collisionality characterizing the long-mean-free-path "banana" regime and briefly explain the terms used.
(b) Sketch what is meant by a "banana orbit" of a trapped particle in the "banana" regime showing the 3D nature of the orbit.

[20 pts.] (a) State (without proof) how the classical (Spitzer) conductivity, $\sigma_{||}$, is modified in the "banana" regime.
(b) Give a simplified (heuristic) estimate for the neoclassical current density in the "banana" regime – specifying the coefficients for the driving forces.
(c) Explain the physics of the "bootstrap current" and give two reasons why it is important.

[20 pts.] (b) Give the trapped-particle flux in the "banana" regime using simplified (heuristic) estimates of the coefficients for these forces (radial density gradient and parallel E-field).

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GENERAL EXAMINATION, PART II

MAY 15, 2012

9 a.m. - 1 p.m.

- Answer all problems.
  Problem 1 has a choice of (A) or (B). Answer only one.
  Problem 5 has a choice of (A) or (B). Answer only one

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

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Problems for Part II, May 15, 2012

1A)	Experimental Methods (Quickie)	
	OR	10 points
1B)	Waves and instabilities (Quickie)	
2)	Irreversible Processes	45 points
3)	Applied Mathematics	40 points
4)	MHD Physics	45 points
5A)	Computational Methods	
	OR	
5B)	Experimental	40 points

Total - 180 points

Part II, Question 1A

Experimental methods "quickie" – 10 pts.

a) Consider an Ohmically heated, large aspect ratio, tokamak plasma. What quantities must be measured in order to determine the global energy confinement time τ_E? What diagnostics would you use to measure these quantities?

b) Now consider a similar tokamak plasma, heated by radial injection of a neutral beam, with $P_{beam} >> P_{Ohmic}$. What additional information would you need for a determination of τ_E? Are there conditions under which your estimate might be prone to error?

Waves and Instabilities

Quickie: Low-frequency limit [10 points]

Assuming one-dimensional nonmagnetized plasma, consider the low-frequency limit of electrostatic perturbations, such that *both* ions and electrons can be considered hot.

(a) [5 points] Write down the dispersion relation for this case and solve it for k.

(b) [5 points] Suppose a boundary condition $E(x = 0) = E_0$. Using the result obtained in (a), find $E(x > 0)$, assuming that plasma is infinite at $x > 0$. What is the physical effect that the solution describes?

In the 3D *guiding-center plasma model* of a very strongly magnetized plasma (gyroradii of both electrons and ions are assumed to be zero), guiding centers stream along a constant magnetic field $\boldsymbol{B}$ while $\boldsymbol{E}\times\boldsymbol{B}$ drifting across the field. ***Calculate the cross-field diffusion coefficient D_x of a test guiding center in a stable, near-thermal-equilibrium plasma by following the steps below.***

Notes:

1. Parts (a), (b), and (c) are independent.
2. You can do part (b) and much of part (d) even if you don't know how to do the other parts.
3. If you do not feel you have time to do the requested mathematics, get partial credit by providing a detailed discussion of what the issues are and a detailed outline of how you would go about attacking this problem.

(a) [12 points]

Show how to derive an approximate expression for D_x in terms of a summation over discrete wave numbers of the electric-field fluctuation spectrum (tensor) $\widehat{\mathsf{C}}_{\boldsymbol{k}}(\omega)$, which is the Fourier transform of $\langle \delta\boldsymbol{E}_{\boldsymbol{k}}(t)\delta\boldsymbol{E}_{\boldsymbol{k}}(t')\rangle$ with respect to $t-t'$. The convention is, for arbitrary function $A(t)$,

$$\widehat{A}(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} dt\, e^{i\omega t} A(t). \tag{1}$$

(b) [5 points]

Given two random functions $\delta A(t)$ and $\delta B(t)$, assuming statistical stationarity, and denoting Fourier transformation with respect to time by a hat, show that

$$\langle \delta\widehat{A}(\omega)\delta\widehat{B}^*(\omega)\rangle = T\widehat{C}(\omega), \tag{2}$$

where T is the total integration time and $\widehat{C}(\omega)$ is the Fourier transform with respect to τ of $\langle \delta A(t+\tau)\delta B^*(t)\rangle$. The result (2) may be useful in part (c).

(c) [18 points]

Find an expression for $\widehat{\mathsf{C}}_{\boldsymbol{k}}(\omega) \stackrel{\text{def}}{=} \langle \delta\boldsymbol{E}_{\boldsymbol{k}}\, \delta\boldsymbol{E}_{\boldsymbol{k}}\rangle(\omega)$ [which enters into part (a)]. If a dielectric function $\mathcal{D}(\boldsymbol{k},\omega)$ enters your formula, give an explicit formula for it. Note that you are dealing with a magnetized plasma here, so $\mathcal{D}(\boldsymbol{k},\omega)$ differs from the one familiar from unmagnetized linear theory.

(Problem continues on next page.)

(d) [10 points]

Combine your results from parts (a) and (c) to get an explicit expression for D_x. Convert the sum over discrete $\boldsymbol{k}$'s to an integral over $\boldsymbol{k}$. Show that you obtain a result of the form

$$D_x \propto \int d\boldsymbol{k}\, \frac{k_\perp^2}{|k_\parallel|\,(k_\perp^2 + k_\parallel^2)^2}\chi(\boldsymbol{k}), \tag{3}$$

where $\chi(\boldsymbol{k})$ is a dimensionless function of $\boldsymbol{k}$ that you should have determined in part (c). Show that the resulting integral [including $\chi(\boldsymbol{k})$] diverges at large $k_\perp$, large $k_\parallel$, and small $k_\parallel$. [If you did not determine $\chi(\boldsymbol{k})$, then (for partial credit) carry on the discussion assuming that $\chi(\boldsymbol{k}) = 1$.] Discuss (thinking deeply) the possibility of curing the divergences by inserting appropriate cutoffs (but don't try to carry out the resulting integrals).

(e) [±10 points] Optional Bonus/Penalty: This calculation differs from others with which you may be familiar in that it contains a *long-wavelength* divergence (at $k_\parallel \to 0$) that, evidently, is not cured by dielectric shielding. State clearly what physics is involved here and describe mathematically what you would do to fix the problem in a way that does not involve a cutoff.

- If you answer this correctly, you will get a *bonus* of up to 10 extra points, depending on the completeness of your answer.
- If you answer it incorrectly, you will be *penalized* at least 5 points and possibly as many as 10 points (so think carefully before trying to answer this optional part).

Part II, Question 3

Applied Mathematics [40 points]

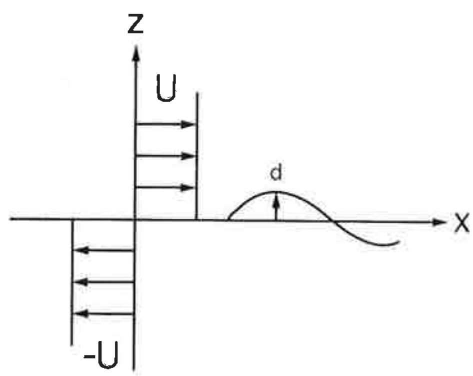
Consider

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-xy=0$$

(a) Find two possible asymptotic behaviors at $x \to +\infty$. (10 pts)
(b) Find two possible asymptotic behaviors at $x \to 0$. (10 pts)
(c) Find two independent integral solutions (20 pts)

MHD Long Problem **45 Points**

In this problem, you will be asked to derive minimum magnetic field to stabilize Kelvin-Helmholtz instability. Suppose that an incompressible, ideal fluid has a flow shear at a sharp boundary at $z = 0$ as shown in the figure where a proper frame is chosen so that $\mathbf{U} = (U, 0, 0)$ at $z > 0$ while $\mathbf{U} = (-U, 0, 0)$ at $z < 0$.



1. (8 minutes) Assuming the perturbation velocity, $\mathbf{v}(x, z)$, is irrotational and in the (x, z) plane, so that a velocity potential, $\phi(x, z)$, can be used:
$$\mathbf{v} = \nabla\phi. \tag{1}$$
When $\phi = \phi_1(z)e^{ik(x-ct)}$ at $z > 0$ and $\phi = \phi_2(z)e^{ik(x-ct)}$ at $z < 0$, derive equations for ϕ_1 and ϕ_2 and find their solutions. Here, k is real and c is complex. Let $A \equiv \phi_1(z \to +0)$ and $B \equiv \phi_2(z \to -0)$.

2. (8 minutes) As shown in the figure, suppose that the boundary perturbation is given by $\xi = de^{ik(x-ct)}$ where d is the amplitude. Use the kinematic boundary condition of
$$v_z = \frac{D\xi}{Dt} \tag{2}$$
for both $z \to +0$ and $z \to -0$ to express A and B in terms of d. Here D/Dt denotes total derivative.

3. (8 minutes) Express perturbed pressure, $p_1 [\equiv p(z \to +0)]$, in terms of A and $p_2 [\equiv p(z \to -0)]$, in terms of B, respectively. Use the dynamic boundary condition,

$$p_1 = p_2, \tag{3}$$

to derive dispersion relation for Kelvin-Helmholtz instability. When is it unstable?

4. (8 minutes) Impose a uniform magnetic field B_0 along the flow direction. Express the perturbed magnetic field in the x direction, B_x, in terms of ϕ for $z > 0$ and for $z < 0$, respectively. [Vector identity: $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$.]

5. (8 minutes) Derive and solve the dispersion relation by adding the perturbed magnetic pressure to the pressure balance,

$$p_1 + \frac{B_{x1} B_0}{\mu_0} = p_2 + \frac{B_{x2} B_0}{\mu_0}, \tag{4}$$

where μ_0 is the vacuum permeability, $B_{x1} \equiv B_x(z \to +0)$, and $B_{x2} \equiv B_x(z \to -0)$. What is the minimum magnetic field to stabilize Kelvin-Helmholtz instability?

Computational Methods (40 points)

Analysis of a Finite Difference Equation:
Consider the scalar one dimensional convection equation, where a is a constant, and U is the unknown function of time and one spatial dimension; $U(t,x)$,

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0 \tag{1}$$

Now consider the following finite difference approximation to Eq. (1):

$$U_{j+1/2}^{n+1} = U_{j+1/2}^{n} - a\frac{\delta t}{\delta x}\left(U_{j+1}^{n+1/2} - U_{j}^{n+1/2}\right) \tag{2}$$

Here, we use the (standard) notation that $U_j^n = U(t^n, x_j)$, where $t^n = n\ \delta t$, and $x_j = j\ \delta x$, with δt and δx being the time step and zone size, respectively. The half-integer subscripts refer to evaluating the quantity halfway between two grid or time points by simple averaging, i.e.,

$$U_{j+1/2}^{n} \equiv \frac{1}{2}\left(U_j^n + U_{j+1}^n\right), \qquad U_j^{n+1/2} = \frac{1}{2}\left(U_j^n + U_j^{n+1}\right) \tag{3}$$

1. (15 points) Show that the finite difference equation (2) is consistent with the partial differential equation (1) in the limit as δt and $\delta x \rightarrow 0$. How does the leading order truncation error scale with δt and δx?

2. (15 points) Use Von Neuman Stability Analysis to calculate the range of values of δt for which the method is stable for different values of a:

3. (10 points) Indicate how you would solve equation (2) to get to time step $n+1$ once the solution at time step n is known?

Experimental 40 Points

1. Consider an interferometer at wavelength, $\lambda = 10.6\ \mu m$ (CO_2 laser), and operates in the presence of spurious vibrations of the optical components. To compensate for these vibrations, interferometry is performed simultaneously using the same optical components at a wavelength of $\lambda = 0.633\ \mu m$ (HeNe laser). The HeNe interferometer is affected much less than the CO_2 interferometer by the plasma phase shift, but still somewhat. If $\omega >> \omega_p$ for both wavelengths:

   (a) Derive an expression for the plasma line integrated density, $\int N_e\, dl$, in terms of the phase shifts, Φ_C and Φ_{He}, of the two interferometers.

   (b) If Φ_{He} can be measured with an accuracy of $+/- \pi$, what uncertainty does this introduce into the plasma density measurement?

   (c) Thus evaluate the fractional error in measuring a 1 m thick plasma of density $10^{14}\ cm^{-3}$, assuming Φ_C is measured exactly.

   (d) Calculate the phase shift for the plasma described in part (c).

Useful expression: $\Phi = K \lambda \int Ne\, dl$, K is a constant.