DEPARTMENT OF ASTROPHYSICAL SCIENCES
PROGRAM IN PLASMA PHYSICS

# GENERAL EXAMINATION, PART I

MAY 13, 2013

9 a.m. - 1 p.m.

- Answer all problems.
  Problem 5 has a choice of (A) or (B). Answer only one.

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, and question # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

- All work on this examination must be independent. No assistance from other persons is permitted.

- No aids (books, calculators, etc.) except for an NRL formulary are permitted.

## Problems for Part I, May 13, 2013

| | | |
|---|---|---|
| 1) | Kinetic Waves (Quickie) | 15 points |
| 2) | Interferometer | 45 points |
| 3) | MHD (Quickie) | 15 points |
| 4) | Irreversible Processes | 50 points |
| 5A) | Math | |
| | **OR** | 45 points |
| 5B) | Computational Plasma Physics | |
| 6) | Particle Orbits (Quickie) | 10 points |

Total - 180 points

Part I, Question 1

**"Kinetic Waves" Quickie – 15 points**

Consider a hot, uniform, unmagnetized plasma in which both the electrons and ions are isotropic, non-drifting Maxwellians. Assume that the plasma is sufficiently dense so that $\omega_{ps}^2 / \omega^2 >> 1$. Derive the dispersion relation for ion acoustic waves, assuming that $T_e >> T_i$.

Useful information:

$$\chi_s = \frac{2\omega_p^2}{k^2 v_t^2}\left\{1 + \frac{\omega - k V_D}{k v_t} Z(\zeta)\right\}$$

where useful expansions for $Z(\zeta)$ can be found in the NRL formulary.

Part I, Question 2
45 points

1. (10) Explain how an interferometer works and what it can measure in a plasma. Draw a diagram of the instrument. Explain some of the limitations with the instrument.

2. (35 pts) Consider an interferometer at wavelength, $\lambda = 10.6\ \mu m$ ($CO_2$ laser), and operates in the presence of spurious vibrations of the optical components. To compensate for these vibrations, interferometry is performed simultaneously using the same optical components at a wavelength of $\lambda = 0.633\ \mu m$ (HeNe laser). The HeNe interferometer is affected much less than the $CO_2$ interferometer by the plasma phase shift, but still somewhat. If $\omega >> \omega_p$ for both wavelengths:

(a) Derive an expression for the plasma line integrated density, $\int N_e\, dl$, in terms of the phase shifts, $\Phi_C$ and $\Phi_{He}$, of the two interferometers.

(b) If $\Phi_{He}$ can be measured with an accuracy of +/- $\pi$, what uncertainty does this introduce into the plasma density measurement?

(c) Thus evaluate the fractional error in measuring a 1 m thick plasma of density $10^{14}$ $cm^{-3}$, assuming $\Phi_C$ is measured exactly.

Useful expression: for a perfect vibration-free interferometer, $\Phi = K \lambda \int N_e\ dl$, where K is a constant.

Part I, Question 3

MHD Quickie (15)

a) What is a flux surface?

b) What is a resonant magnetic perturbation (RMP)?

c) Explain why a relatively small RMP can have a significant effect on the corresponding rational surface. You can discuss this in the context of a large aspect ratio torus with circular cross-section, which can be treated as a cylinder with periodic boundary conditions at the ends.

# Irreversible Processes [50 points]

According to Braginskii's discussion of classical transport, in the limit where $\nu_e/\omega_{ce} \ll 1$ the perpendicular friction force $\boldsymbol{R}_\perp$ is

$$\boldsymbol{R}_\perp = -[(mn)_e/\tau_e](\boldsymbol{u}_{\perp,e} - \boldsymbol{u}_{\perp,i}), \tag{1}$$

with Braginskii's specific definition of $\tau_e$. **Derive this result quantitatively by performing a direct kinetic calculation of the perpendicular electron current driven by an imposed electric field $\boldsymbol{E}_\perp$.** You may assume that the ions are very massive.

Hints:

1. Ultimately you will find that one piece of the perpendicular driven current obeys $\boldsymbol{j}_\perp = \Sigma_\perp \boldsymbol{E}_\perp$ for some $\Sigma_\perp$. This is to be compared with the alternate way that Braginskii sometimes writes Eq. (1), $\boldsymbol{R}_\perp = ne\boldsymbol{j}_\perp/\sigma_\perp$. You may find that $\Sigma_\perp \neq \sigma_\perp$. If they differ, you will need to proceed further and show how to obtain $\sigma_\perp$ from $\Sigma_\perp$.

2. Exploit $\nu_e/\omega_{ce} \ll 1$ and solve the problem by perturbation theory. In the algebra, it may be helpful to recognize that

$$\boldsymbol{v} \times \widehat{\boldsymbol{b}} \cdot \frac{\partial}{\partial \boldsymbol{v}} = -\frac{\partial}{\partial \zeta}, \tag{2}$$

where $\zeta$ is the cylindrical-polar angle in velocity space.

3. You may not have time to complete all of the algebra. Demonstrating that you understand what to do is more important than getting all of the numbers correct. You may want to begin by giving a detailed outline of how you will attack the problem.

## Part I, Question 5A – 45 points

Consider

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy = 0$$

a) Find two possible asymptotic behaviors at $x \to 0$
b) Find two possible asymptotic behaviors at $x \to +\infty$
c) Find a Fourier-Laplace integral representation for the solution
d) Find an integration contour if $y(0) = 1$.

**Part I, Question 5B**

**Computational Plasma Physics**
45 points

**Electron Hall MHD**

Consider the 1-dimensional electron Hall-MHD equation for the time evolution of the *(x,y)* Cartesian components of the magnetic field, which vary in the z-dimension:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left[ \alpha (\nabla \times \mathbf{B}) \times \mathbf{B} \right] \tag{1.0}$$

Here the magnetic field is given by:

$$\mathbf{B} \equiv b_x(z)\hat{\mathbf{x}} + b_y(z)\hat{\mathbf{y}} + B_0\hat{\mathbf{z}} \tag{2.0}$$

with $B_0$ and $\alpha$ being real constants.

1. (15 points) Find the $2 \times 2$ matrix **A** for which the linearized form of equations(1-2) can be written as:

$$\frac{\partial}{\partial t}\begin{bmatrix} b_x \\ b_y \end{bmatrix} = \alpha B_0 \mathbf{A} \cdot \frac{\partial^2}{\partial z^2}\begin{bmatrix} b_x \\ b_y \end{bmatrix} \tag{3.0}$$

2. (15 points) Write down a FTCS (forward in time, centered in space) finite difference method for solving equation (3.0), and analyze the stability of the method using the Von Neumann stability analysis technique.

3. (15 points) Suggest an alternative finite difference method for solving equation (3.0) with improved stability properties to that of part 2, and give the condition on the time-step for stability.

## Particle Orbits (Quickie) [10 Points]

Consider a magnetic mirror device where $B_{\mathrm{min}}$ and $B_{\mathrm{max}}$ denote the minimum and maximum magnetic field. A group of isotropically-distributed particles are released at the magnetic minimum. What fraction of these particles will be trapped by the mirror?

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# GENERAL EXAMINATION, PART II

MAY 14, 2013

9 a.m. - 1 p.m.

- Answer all problems.
  Problem 4 has a choice of (A) or (B). Answer only one.

- The exam has been designed to require about 3 hours of work; however we have allowed you an extra hour. Thus the total time allotted for this day is 4 hours. Scores on questions will be weighted in proportion to their allotted time.

- Start each numbered problem in a new test booklet. Put your name, question # and part # on every booklet title page.

- When you do not have time to put answers into forms that satisfy you, indicate specifically how you would proceed if more time were available. If you do not attempt a particular problem, write on a sheet of paper "I have not attempted Problem _____ " and sign your name.

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## Problems for Part II, May 14, 2013

| | | |
|---|---|---|
| 1) | Neoclassical Transport & Drift Waves | 60 points |
| 2) | Waves & Instabilities | 60 points |
| 3) | MHD | 45 points |
| 4A) | Experimental Plasma Physics (quickie) | |
| | **OR** | 15 points |
| 4B) | Math | |
| | | Total - 180 points |

## Neoclassical Transport & Drift Waves [60 points]

This problem has two parts: Part 1 deals with the "bootstrap current" in a toroidal plasma with driving forces being a scalar pressure gradient in the radial direction (assume no temperature gradient). Part 2 deals with a drift wave instability.

Part 1: Give a simplified (heuristic) estimate for the bootstrap current in the long-mean-free-path "banana" regime.

[5 pts.] (a) What is the banana regime collisionality criterion?
[20 pts.] (b) Heuristically estimate the charge flux (current density) in the banana regime.
[5 pts.] (c) *Describe* (without actual calculations) how this bootstrap current is formally derived.
[5 pts.] (d) How do trapped particles contribute to the bootstrap current?
[5 pts.] (e) Why is the bootstrap current important in magnetically-confined plasma systems?

**Part 2**: Ion drift waves driven unstable by ion temperature gradients (ITG or "eta-i" modes) is most simply obtained from the local dispersion relation given by:

$$1-\frac{\omega_{*e}}{\omega}+b_s-\frac{k_{\parallel}^2 c_s^2}{2\omega^2}\left(1-\frac{\omega_{*pi}}{\omega}\right)=0$$

Here: $\omega_{*pi}=\omega_{*i}\left[1+\eta_i\right]$ with "eta-i" being the ratio of density to ion temperature gradient scale lengths. To obtain this result, the adiabatic response for electrons,

$n_e/n_0=|e|\phi/T_e$, together with a fluid derivation for the ion response was used.

[15 pts.] (a) Derive the ITG-instability in the limit of very large "eta-i."

[5 pts.] (b) Justify the use of the "quasi-neutrality condition" relating the perturbed ion and electron density responses that gives the dispersion relation.

## Waves and Instabilities

### Weibel instability [60 points]

In plasmas with nonisotropic temperature, transverse waves can be subject to the so-called Weibel instability (Weibel, 1959). Below, you are asked to calculate the effect in the simplest case of **nonmagnetized** nonrelativistic collisionless plasma with motionless ions.

(a) [25 points] Accounting for the nonzero magnetic field **of the wave**, show that the dielectric tensor in such a plasma can be expressed as

$$\epsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij}\left(1 - \frac{\omega_p^2}{\omega^2}\right) + \frac{\omega_p^2}{\omega^2}\int \frac{k v_i v_j}{\omega - k v_z}\frac{\partial f_0}{\partial v_z}\, d^3 v, \tag{1}$$

where the wave vector $\mathbf{k}$ is assumed parallel to $z$ axis, and $\int f_0(\mathbf{v})\, d^3 v = 1$.

(b) [15 points] Assume now that $f_0(\mathbf{v})$ is bi-Maxwellian with zero average velocity; namely,

$$f_0(\mathbf{v}) = \frac{1}{\pi^{3/2}\, w_\perp^2 w_\parallel} \exp\left(-\frac{v_x^2}{w_\perp^2} - \frac{v_y^2}{w_\perp^2} - \frac{v_z^2}{w_\parallel^2}\right), \tag{2}$$

where $w_\perp^2 = 2T_\perp/m$, and $w_\parallel^2 = 2T_\parallel/m$. Show that $\epsilon_{ij}$ is diagonal. Then calculate $\epsilon_\perp \equiv \epsilon_{xx} = \epsilon_{yy}$ explicitly and show that the dispersion relation of transverse waves can be written as follows:

$$\omega^2 - k^2 c^2 - \omega_p^2 + \omega_p^2\left(\frac{T_\perp}{T_\parallel}\right)[1 + \zeta Z(\zeta)] = 0. \tag{3}$$

[The plasma dispersion function, $Z(\zeta)$, can be found in the NRL Plasma Formulary.]

(c) [10 points] For hot plasma, show that Eq. (3) is approximately linear in $\omega$ and derive the corresponding $\omega(k)$ explicitly. (For simplicity, assume $k > 0$.) Show that, at $T_\perp > T_\parallel$, there always exist $k$ for which waves are unstable.

(d) [10 points] For cold plasma, retain thermal corrections to the **lowest nonvanishing** order and show that Eq. (3) is then biquadratic in $\omega$ (i.e., quadratic in $\omega^2$). Calculate $\omega(k)$ to the first order in temperature and show that one of the modes is unstable. Show that, to justify the cold-plasma approximation for this mode, $T_\perp \gg T_\parallel$ is required.

## MHD Long Problem (Total 45 Minutes)

In this problem, you will be asked to derive minimum energy states of an MHD system using variational principle under various constraints. Under proper normalizations, three global quantities of an incompressible MHD system, total energy $W$, total magnetic helicity $H$, and total cross helicity $K$ are given by

$$W = \frac{1}{2}\int(V^2 + B^2)d\tau \qquad (1)$$

$$H = \frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{B}d\tau \qquad (2)$$

$$K = \int \boldsymbol{V}\cdot\boldsymbol{B}d\tau \qquad (3)$$

where $\boldsymbol{V}$ and $\boldsymbol{B}$ are (properly normalized) velocity and magnetic fields, respectively, and $\int d\tau$ is an integral over a volume, $\tau$. $\boldsymbol{A}$ is the vector potential so that $\nabla\times\boldsymbol{A}\equiv\boldsymbol{B}$.

All these three quantities are strictly conserved in the ideal limit. In realistic plasmas where dissipations are finite, however, these quantities decay.

1. (10 min) If $W$ decays much faster than $K$, the system tends to evolve towards a minimum energy state while conserving $K$. Find such minimum energy states by seeking conditions to satisfy $\delta(W-\mu K)=0$ for arbitrary variations in $\boldsymbol{V}$ and $\boldsymbol{B}$, $\delta\boldsymbol{V}$ and $\delta\boldsymbol{B}$, which can be assumed to vanish on the boundary of $\tau$. Here $\mu$ is a Lagrange multiplier. What value(s) should $\mu$ take? How is $\boldsymbol{V}$ related to $\boldsymbol{B}$?

2. (15 min) Replace $K$ with $H$ in the problem (1) to solve $\delta(W-\lambda H)=0$ where $\lambda$ is a second Lagrange multiplier. What are the properties of $\boldsymbol{B}$? [Vector identity: $\nabla\cdot(\delta\boldsymbol{A}\times\boldsymbol{A}) = \boldsymbol{A}\cdot\nabla\times\delta\boldsymbol{A} - \delta\boldsymbol{A}\cdot\nabla\times\boldsymbol{A}$.]

3. (10 min) Repeat the calculation for the case where $W$ decays much faster than both $K$ and $H$. How is $\boldsymbol{V}$ related to $\boldsymbol{B}$? What are the properties of $\boldsymbol{V}$ and $\boldsymbol{B}$?

4. (10 min) In the problem (3), show that the answers of the problems (1) and (2) can be recovered by taking $\lambda\to 0$ and $\mu\to 0$, *respectively*. What if we take $\lambda\to 0$ and $\mu\to 0$, *simultaneously*?

Part II, Question 4A
15 points

Experimental plasma physics

You have been asked to design an experiment for a space shuttle mission. The experiment involves two shuttles in very close orbit connected by a long insulated cable. A differential voltage bias will be applied between the shuttles and the current collected will be monitored as a function of the applied bias.

The mission will be flown at 200 km altitude, near the lower boundary of the F layer, where the local electron density is expected to be between $10^5$ and $10^6$ $cm^{-3}$. Since the plasma in this region is primarily generated by photoionization, electrons of 1 – 2 eV are expected, and the ionic species is predominantly singly ionized cold oxygen. The earth's magnetic field can be neglected (by declaration). The conducting (collection) area of each shuttle is 20 $m^2$.

(a) Calculate the voltage and current requirements for the power supply which provides the bias.
(b) Sketch the current-voltage characteristic which you would expect if the electron density were $10^6$ $cm^{-3}$, and the electron temperature 2 eV.

## Part II, Question 4B – 15 points

Consider

$$\frac{d^2\psi}{dx^2} + Q(x)\psi = 0$$

a) If $Q(x)$ is real and finite show there exists a conserved flux for this equation.

b) If $Q(x)$ has a first order pole, $Q = a/x$, with $a > 0$, show how the flux change across the singularity implies the Landau prescription.