

DEPARTMENT OF ASTROPHYSICAL SCIENCES,  
PROGRAM IN PLASMA PHYSICS  
GENERAL EXAMINATION, PART I

May 12, 2014

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name and the question number on the title page of each booklet.
- When you do not have time to put answers into forms that satisfy you, indicate *specifically* how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem \_\_\_\_" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

**Contents**

Page

|                                       |   |
|---------------------------------------|---|
| I.1: Mirror Confinement [40 points]   | 2 |
| I.2: Geometrical Optics [15 points]   | 3 |
| I.3: MHD [50 points]                  | 4 |
| I.4: Experimental Methods [10 points] | 5 |
| I.5: Neoclassical Theory [35 points]  | 6 |
| I.6: Applied Mathematics [30 points]  | 9 |

## I.1: Mirror Confinement [40 points]

Consider a magneto-electric particle trap in the region  $-L < z < L$ . To accomplish this trap, suppose that there is a magnetic field in the  $z$  direction such that

$$B = \begin{cases} B_0 \left[ 1 + (R-1) \left( \frac{z^2}{L^2} \right) \right] & \text{if } -L < z < 0, \\ B_0, & \text{if } z \geq 0. \end{cases}$$

Here the mirror ratio  $R$  is a constant greater than one. Suppose that there is also an electric potential

$$\phi = \begin{cases} 0 & \text{if } z < 0, \\ \phi_0 \left( \frac{z^2}{L^2} \right) & \text{if } 0 \leq z < L, \\ \phi_0 & \text{if } z > L. \end{cases}$$

(a) [5 points] Describe how ions might be trapped in this configuration of magnetic and electric fields. Would electrons also be trapped in the same fields?

(b) [10 points] Derive a trapping condition for confined particles in terms of the particle midplane perpendicular energy  $W_{\perp 0}$  and midplane parallel energy  $W_{\parallel 0}$ , where these energies are defined at the axial location  $z = 0$ .

(c) [5 points] Sketch the trapping condition in  $W_{\perp 0}$ - $W_{\parallel 0}$  space. If trapped ions of charge state  $q$  were scattered in pitch angle but not in energy through collisions, from what end of the device would they leave? Does this answer depend on the ion energy? Please explain very briefly (in one sentence).

(d) [10 points] Suppose now that both the electric potential and the magnetic field are slowly varying functions of time. Show that the second adiabatic invariant can be put into the form

$$\dot{W}_{\parallel 0}^{1/2} (z_M + z_E) = \text{const.}$$

Here  $z_M$  and  $z_E$  are the turning points in the regions  $z < 0$  and  $z > 0$ , respectively.

You may wish to use (but you do not really need) the integral

$$\int_0^1 ds (1 - s^2)^{1/2} = \frac{\pi}{4}.$$

(e) [5 points] Suppose that the potential  $\phi_0(t)$  is slowly changing in time, such that  $\phi_0(t = T) = \gamma \phi_0(t = 0)$ . Suppose that the magnetic field  $B_0(t)$  is slowly changing in time, such that  $B_0(t = T) = \alpha B_0(t = 0)$ . Write down the condition that at time  $T$  a particle initially trapped will become untrapped, leaving the trap on the side  $z > 0$ .

(f) [5 points] For a trapped particle with initial energies  $W_{\perp 0} \gg W_{\parallel 0}$  and  $W_{\perp 0} \gg q\phi_0$ , give a prescription for removing the particle on the side  $z < 0$  without any collisions taking place and by changing only  $\phi_0$  and  $B_0$ .

## I.2: Geometrical Optics [15 points]

Consider electron whistler waves traveling in a cold, stationary, collisionless plasma along a  $z$  axis that is parallel to a static magnetic field. Assume that the index of refraction is large ( $n \gg 1$ ), and that the wave frequency  $\omega$  is much less than the electron cyclotron frequency  $|\Omega_e|$ . Suppose that both  $\Omega_e$  and the plasma density depend on  $z$  slowly, such that the geometrical-optics approximation is satisfied.

(a) [5 points] Show that the dynamics of an envelope of such waves is similar to the dynamics of a *nonrelativistic* particle with a  $z$ -dependent mass. Write down the corresponding dynamical equations explicitly.

(b) [10 points] For a stationary whistler wave, calculate how its electric-field amplitude evolves along  $z$ .

### I.3: MHD [50 points]

In this problem, you will be asked to derive the condition for entropy production through a perpendicular MHD shock.

(a) [10 points] From the ideal MHD conservation laws

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (1)$$

$$\frac{\partial}{\partial t} (\rho \mathbf{V}) + \nabla \cdot \left[ \rho \mathbf{V} \mathbf{V} + \left( P + \frac{B^2}{2\mu_0} \right) \mathbf{I} - \frac{\mathbf{B} \mathbf{B}}{\mu_0} \right] = 0, \quad (2)$$

$$\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} + \frac{B^2}{2\mu_0} + \frac{P}{\gamma - 1} \right) + \nabla \cdot \left[ \left( \frac{\rho V^2}{2} + \frac{\gamma P}{\gamma - 1} \right) \mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right] = 0, \quad (3)$$

derive the Rankine-Hugoniot relations for stationary perpendicular MHD shocks where the magnetic field is *parallel* to the shock front:

$$\rho_2 V_2 = \rho_1 V_1, \quad (4)$$

$$P_2 + \rho_2 V_2^2 + \frac{B_2^2}{2\mu_0} = P_1 + \rho_1 V_1^2 + \frac{B_1^2}{2\mu_0}, \quad (5)$$

$$\left( \frac{\gamma}{\gamma - 1} P_2 + \frac{\rho_2 V_2^2}{2} + \frac{B_2^2}{\mu_0} \right) V_2 = \left( \frac{\gamma}{\gamma - 1} P_1 + \frac{\rho_1 V_1^2}{2} + \frac{B_1^2}{\mu_0} \right) V_1, \quad (6)$$

$$B_2 V_2 = B_1 V_1. \quad (7)$$

Here  $\mathbf{V}$  is the velocity field,  $\mathbf{B}$  is the magnetic field,  $P$  is the plasma pressure,  $\rho$  is the density, and  $\gamma$  is the ratio of specific heats. The subscript 1 (2) denotes quantities upstream (downstream) of the shock.

(b) [15 points] Upon defining  $\rho_2/\rho_1 = X$ , one has  $V_2/V_1 = 1/X$  and  $B_2/B_1 = X$ . Show that  $P_2/P_1$  is given by both of the following equations:

$$\frac{P_2}{P_1} = \gamma \left( 1 + \frac{2}{\gamma \beta_1} \right) \left( 1 - \frac{1}{X} \right) M^2 + \frac{1}{\beta_1} (1 - X^2) + 1, \quad (8)$$

$$\frac{P_2}{P_1} = \frac{\gamma - 1}{2} \left( 1 + \frac{2}{\gamma \beta_1} \right) \left( X - \frac{1}{X} \right) M^2 + \frac{\gamma - 1}{\gamma} \frac{2}{\beta_1} (X - X^2) + X, \quad (9)$$

where the Mach number is defined by  $M \equiv V_1/\sqrt{V_S^2 + V_A^2}$ . Here the sound speed  $V_S \equiv \sqrt{\gamma P_1/\rho_1}$  and the Alfvén speed  $V_A \equiv B_1/\sqrt{\mu_0 \rho_1}$ . The upstream beta is defined as  $\beta_1 \equiv P_1/(B_1^2/2\mu_0)$ .

(c) [5 points] Show that  $X \rightarrow (\gamma + 1)/(\gamma - 1)$  as  $M \rightarrow \infty$  regardless of the value of  $\beta_1$ .

(d) [20 points] Derive the condition for entropy, defined here as  $s \equiv \ln(P/\rho^\gamma)$ , to increase across the shock so that  $s_2 - s_1 > 0$ . Discuss its physical meanings.

#### I.4: Experimental Methods [10 points]

(a) [5 points] Consider an Ohmically heated, *large*-aspect-ratio tokamak plasma. What quantities must be measured in order to determine the global energy confinement time  $\tau_E$ ? What diagnostics would you use to measure these quantities?

(b) [5 points] Now consider a similar tokamak plasma, heated by radial injection of a neutral beam with  $P_{\text{beam}} \gg P_{\text{Ohmic}}$ . What additional information would you need for a determination of  $\tau_E$ ? Under what conditions will your measurement be accurate?

## I.5: Neoclassical Theory [35 points]

Consider the simplified ion temperature equation

$$\frac{3}{2}n\frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q}. \quad (1)$$

(Ion subscripts will consistently be omitted.) In a collisional plasma, according to Braginskii, the ion heat flux is (in the limit  $\delta \stackrel{\text{def}}{=} \nu/\omega_c \ll 1$  and for  $Z = 1$ )

$$\mathbf{q} = \mathbf{q}_{\parallel} + \mathbf{q}_{\times} + \mathbf{q}_{\perp}, \quad (2)$$

where

$$\mathbf{q}_{\parallel} = -n\kappa_{\parallel}\nabla_{\parallel}T, \quad (3a)$$

$$\mathbf{q}_{\times} = \frac{5}{2}n\left(\frac{cT}{eB}\right)\hat{\mathbf{b}} \times \nabla T, \quad (3b)$$

$$\mathbf{q}_{\perp} = -n\kappa_{\perp}\nabla_{\perp}T, \quad (3c)$$

with

$$\kappa_{\parallel} = 3.9v_i^2/\nu, \quad (4a)$$

$$\kappa_{\perp} = 2\rho^2\nu, \quad (4b)$$

where  $\nu \stackrel{\text{def}}{=} \tau^{-1}$  is the inverse of Braginskii's collision time. In a torus (assumed here to be axisymmetric), the cross-field heat flux is enhanced over the classical value  $-n\kappa_{\perp}\partial T/\partial r$  by a Pfirsch-Schlüter-like effect. **Discuss the calculation of the Pfirsch-Schlüter enhancement to heat flux as follows.**

**Note:** The five parts of this problem are largely independent.

**(a) [9 points]** Discuss why there should be an enhancement. Give a step-by-step outline of how one should proceed to do this calculation. At the very least, your outline should cover what to do at each of the orders  $\delta^0$ ,  $\delta^1$ , and  $\delta^2$ , and there may be multiple steps at each order. Do not do any mathematical calculations in this part.

**Note:** This calculation does not involve any considerations relating to an electric field, so do not invoke that in your discussion.

(Problem continues on next page.)

Consider the  $\{r, \theta, \varphi\}$  coordinate system shown in Fig. 1, and assume the model magnetic field  $\mathbf{B} = \mathbf{B}_\theta + \mathbf{B}_\varphi$  with

$$\mathbf{B}_\theta = \frac{\epsilon(r)}{q(r)} \mathbf{B}_\varphi, \quad \mathbf{B}_\varphi = \frac{B_0 R_0}{R} = \frac{B_0}{1 + \epsilon(r) \cos \theta}, \quad (5)$$

and  $\epsilon(r) \stackrel{\text{def}}{=} r/R_0$ . Turn your above outline into mathematics and calculate the enhanced heat flux as follows.

- If necessary, you may assume an equilibrium with a scalar pressure.
- Assume that  $\epsilon \ll 1$  (small inverse aspect ratio).
- It can be shown that
  - $T$  is a flux function to zeroth order in  $\delta$ ;
  - $\nabla \cdot \mathbf{q}_\times = -\mathbf{q}_\times \cdot \nabla \ln B^2$ .

You may assume that these results are known.

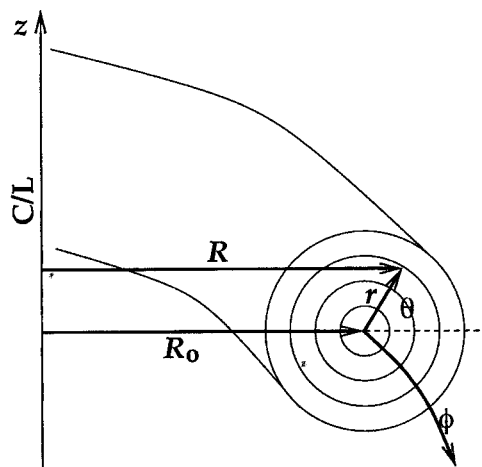


FIG. 1 Coordinate system for model field with concentric circular flux surfaces.  $R = R_0 + r \cos \theta = R_0(1 + \epsilon \cos \theta)$ .

(b) [7 points] At first order in  $\delta$ , it is convenient to write  $q_{\parallel}^{(1)} = \alpha^{(1)} B$ . Show that  $\alpha^{(1)}$  obeys the magnetic differential equation

$$\mathbf{B} \cdot \nabla \alpha^{(1)} = \mathbf{q}_\times \cdot \nabla \ln B^2. \quad (6)$$

Solve that equation and show that

$$\alpha^{(1)} = -\frac{5}{2} \left( \frac{q}{\epsilon} \right) \left( \frac{ncT}{eB^2} \right) T' + C(r), \quad (7)$$

where  $C$  is an undetermined constant of integration.

(Problem continues on next page.)

(c) [7 points] Show that

$$\alpha^{(1)} = -\frac{5}{2} \left( \frac{q}{\epsilon} \right) \left( \frac{ncT}{e} \right) T' \left( \frac{1}{B^2} - \frac{1}{\langle B^2 \rangle} \right). \quad (8)$$

(d) [7 points] Show that the radial component of the flux is

$$\hat{\mathbf{r}} \cdot \mathbf{q}_x^{(1)} = -\frac{25}{4} n \left( \frac{cT}{e} \right)^2 \left( \frac{q^2}{\epsilon^2} \right) \frac{1}{\kappa_{\parallel}} \left( \frac{1}{B^2} - \frac{1}{\langle B^2 \rangle} \right) T'. \quad (9)$$

(e) [5 points] What remains is to average the radial heat flux over a flux surface. Show that

$$\left\langle \frac{1}{B^2} \right\rangle - \frac{1}{\langle B^2 \rangle} \approx \frac{2\epsilon^2}{B_0^2}. \quad (10)$$

---

Although you don't need to do it, when one puts it all together one finds that

$$\langle q^r \rangle = -n\kappa_{\perp} T' (1 + 1.6q^2). \quad (11)$$

Possibly useful information:

- Given three generalized coordinates  $z^i$  ( $i = 1, 2, 3$ ), the volume element is

$$d\mathbf{x} = dz^1 dz^2 dz^3 J, \quad (12)$$

which defines the Jacobian  $J$ :  $J = |\partial(\mathbf{x})/\partial(\mathbf{z})|$ .

- If  $\psi$  is a flux-surface label, the flux-surface average of some scalar function  $F(\psi, \theta, \varphi)$  is

$$\langle F \rangle(\psi) = \frac{\int_0^{2\pi} \int_0^{2\pi} d\theta d\varphi J F}{\int_0^{2\pi} \int_0^{2\pi} d\theta d\varphi J}. \quad (13)$$

- The flux-surface average of a divergence is

$$\langle \nabla \cdot \mathbf{A} \rangle = \frac{1}{V'} \frac{\partial}{\partial \psi} (V' \langle A^{\psi} \rangle), \quad (14)$$

where  $V'$  is the denominator of Eq. (13) and  $A^i \stackrel{\text{def}}{=} \mathbf{A} \cdot \nabla z^i$  are the contravariant components of the vector  $\mathbf{A}$ .

- The ratio of the parallel and perpendicular Braginskii resistivities is 0.51.

## I.6: Applied Mathematics [30 points]

Consider the boundary-layer treatment of the differential equation

$$\epsilon \frac{d^2 y}{dx^2} + x^3 \frac{dy}{dx} - 2xy = 0 \quad (1)$$

with the boundary conditions  $y(0) = y(1) = 1$ . Assume  $0 < \epsilon \ll 1$ .

- (a) [5 points] Evaluate the width  $\delta$  of the layer in terms of  $\epsilon$ .
- (b) [5 points] Find the outer-layer equation and the leading-order solution in  $\epsilon$ .
- (c) [5 points] Find the inner-layer equation. Working only to leading order in  $\epsilon$ , examine (again only) the dominant asymptotic behavior of the solution near the matching boundary (i.e., when the rescaled length  $X$  for the inner layer is large and positive).
- (d) [5 points] Show that the integral representation of the leading-order inner-layer solution can be given by

$$I_k(z) = \int_{C_k} e^{zt-t^3/6} dt, \quad (2)$$

with the proper contour path  $C_k$  ( $k = 1, 2, 3$ ) in the complex  $t$  domain, as shown in Fig. 2.

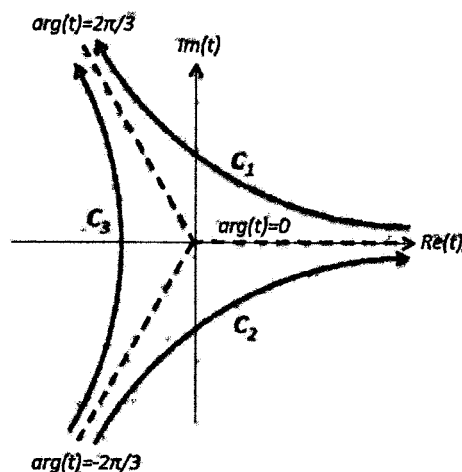


FIG. 2 Integration paths  $C_k$  ( $k = 1, 2, 3$ ) in the complex  $t$  domain. The angle of each end point is indicated by  $\arg(t)$ .

- (e) [5 points] Which path, or combination of them, gives the relevant solution for the matching condition?
- (f) [5 points] Find the leading-order inner-layer and uniform solutions by using the integral representation.

DEPARTMENT OF ASTROPHYSICAL SCIENCES,  
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

May 13, 2014

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name and the question number on the title page of each booklet.
- When you do not have time to put answers into forms that satisfy you, indicate *specifically* how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem \_\_\_\_" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

**Contents**

Page

|   |   |
|---|---|
| II.1: Experimental Methods [40 points] . . . . .    | 2 |
| II.2: MHD [15 points] . . . . .                     | 3 |
| II.3: Irreversible Processes [45 points] . . . . .  | 4 |
| II.4: General Phenomena [10 points] . . . . .       | 6 |
| II.5: Waves and Instabilities [50 points] . . . . . | 7 |
| II.6: Computational Physics [20 points] . . . . .   | 9 |

## II.1: Experimental Methods [40 points]

In the absence of secondary electron emission, the condition that a probe or other particle-collecting surface immersed in a plasma be at the floating potential (i.e., conduct no current to ground) is satisfied when the electron and ion currents are equal, or  $j_i = j_e$ :

$$n_e \left( \frac{kT_e}{2\pi m_e} \right)^{1/2} \exp \left( \frac{-e(V_{sp} - V_f)}{kT_e} \right) = 0.6n_e \left( \frac{kT_e}{m_i} \right)^{1/2}.$$

- (a) [5 points] What modification to the above relationship is introduced if the secondary-electron emission coefficient of the collecting surface is nonzero?
- (b) [15 points] Describe the qualitative effect of a *moderate* secondary-electron emission coefficient on the floating potential. Consider  $\delta_e = 0.5$  as a quantitative example, and solve for the modified floating potential.
- (c) [10 points] For what secondary-electron emission coefficient would the particle-collecting surface (or probe) float at approximately the space potential? If it develops that no material with such a secondary-electron emission coefficient is available, can you think of another approach to probe construction which would allow a probe of some type to float at the space potential? Or imagine a probe construction that would allow you to at least approximately determine the space potential?
- (d) [5 points] If the particle-collecting surface were a conducting limiter to which a voltage bias could be applied, what effect would biasing the limiter at the space potential have on the total power flowing to the limiter?
- (e) [5 points] If the conducting limiter were replaced with an insulating limiter (obviously not biased in this case), would there be an effect on the power flow to the limiter? Can you say anything about the potential at the limiter surface? Compare to an unbiased limiter, and to the situation in part (d) above.

## II.2: MHD [15 points]

The Sun is the source of the solar wind, which permeates the heliosphere. Assume that the Sun is rotating about its axis with a constant angular speed  $\omega_s$ , and that the solar wind flows out purely radially with a constant speed  $V_{sw}$ . In a spherical-polar coordinate system  $(r, \theta, \phi)$  and assuming that the magnetic field is initially purely radial and “frozen” into the plasma, show that the magnetic field *in the equatorial plane* of the Sun is wound up by the solar wind in the shape of a spiral (the so-called Archimedean spiral), given by

$$\phi = \left( \frac{\omega_s}{V_{sw}} \right) r + \phi_0,$$

where  $\phi$  is an azimuthal angle measured from the polar axis and  $\phi_0$  is a constant reference angle. Obtain an expression for  $B_r/B_\phi$  in the equatorial plane.

### II.3: Irreversible Processes [45 points]

In Braginskii's derivation of collisional transport equations and coefficients, he works with the usual kinetic equation written in the particle coordinates  $\{\mathbf{x}, \mathbf{v}\}$ . However, for cross-field effects it is often advantageous to use gyrocenter coordinates instead. This problem addresses the calculation of cross-field density diffusion using the latter approach.

(a) [5 points] The kinetic equation contains the velocity derivative  $\partial/\partial \mathbf{v}$  evaluated with  $\mathbf{x}$  held constant. Instead, consider the lowest-order gyrocenter variables  $\{\mathbf{X}, \mathbf{V}\}$ , where

$$\mathbf{X} \stackrel{\text{def}}{=} \mathbf{x} - \boldsymbol{\rho}, \quad (1a)$$

$$\mathbf{V} \stackrel{\text{def}}{=} \mathbf{v}, \quad (1b)$$

and

$$\boldsymbol{\rho} \stackrel{\text{def}}{=} \hat{\mathbf{b}} \times \mathbf{v}_\perp / \omega_c. \quad (2)$$

Assume that  $\mathbf{B} = B_0 \hat{\mathbf{b}} = \text{const.}$  Show that

$$\left. \frac{\partial}{\partial \mathbf{v}} \right|_{\mathbf{x}} = \omega_c^{-1} \hat{\mathbf{b}} \times \left. \frac{\partial}{\partial \mathbf{X}} \right|_{\mathbf{V}} + \left. \frac{\partial}{\partial \mathbf{V}} \right|_{\mathbf{X}}. \quad (3)$$

*Please be sure to adequately distinguish upper case from lower case, or use a different notation like  $\{\overline{\mathbf{X}}, \overline{\mathbf{V}}\}$  for the upper-case symbols.*

(b) [5 points] Write the Landau operator in the above gyrocenter variables (and call it  $\tilde{C}$ ).

(c) [5 points] The distribution  $\tilde{F}_s(\mathbf{X}, \mathbf{V}, t)$  is the same object as the particle distribution  $f_s(\mathbf{x}, \mathbf{v}, t)$ , but expressed in different variables. Show that  $\tilde{F}_s$  obeys

$$\frac{\partial \tilde{F}_s(\mathbf{X}, \mathbf{V})}{\partial t} + \mathbf{V}_\parallel \nabla_\parallel \tilde{F}_s + \mathbf{V}_E \cdot \nabla \tilde{F}_s + \left( \frac{q}{m} \right)_s \mathbf{E} \cdot \frac{\partial \tilde{F}_s}{\partial \mathbf{V}} + \omega_{cs} \mathbf{V} \times \hat{\mathbf{b}} \cdot \frac{\partial \tilde{F}_s}{\partial \mathbf{V}} = -\tilde{C}_s[\tilde{F}], \quad (4)$$

where  $\nabla \stackrel{\text{def}}{=} \partial/\partial \mathbf{X}$  and  $\mathbf{V}_E \stackrel{\text{def}}{=} c\mathbf{E} \times \hat{\mathbf{b}}/B$ . By integrating this equation over  $\mathbf{V}$ , find an evolution equation for the gyrocenter density  $N_s(\mathbf{X}, t)$ .

*(Problem continues on next page.)*

In a collisional regime, the lowest-order distribution is a local Maxwellian  $f_{\text{IM},s}(\mathbf{x}, \mathbf{v}, t)$  in the particle coordinates. This is annihilated by the collision operator (no matter in which coordinates that operator is written). Assume that the  $\mathbf{x}$  dependence of that local Maxwellian is only through density, i.e.,  $f_{\text{IM},s}(\mathbf{x}, \mathbf{v}, t) = [n_s(\mathbf{x}, t)/\bar{n}_s] f_{\text{M},s}(\mathbf{v})$ , where  $f_{\text{M}}(\mathbf{v})$  is an absolute Maxwellian. One has  $f_{\text{IM},s}(\mathbf{x}, \mathbf{v}, t) = \tilde{F}_{\text{IM},s}(\mathbf{X}, \mathbf{V}, t)$ . To repeat, it is true that  $\tilde{C}_s[\tilde{F}_{\text{IM}}] = 0$ . However, consider the gyroaveraged distribution  $\bar{F}_s(\mathbf{X}, \mathbf{V}, t) \stackrel{\text{def}}{=} \langle \tilde{F}_s(\mathbf{X}, \mathbf{V}, t) \rangle$ , where the average is over gyrophase. This quantity is not annihilated by the collision operator:  $\tilde{C}_s[\bar{F}] \neq 0$ .

(d) [30 points] To lowest order in the gyroradius, one has

$$\bar{F}_{\text{IM},s}(\mathbf{X}, \mathbf{V}, t) = [N_s(\mathbf{X}, t)/\bar{n}_s] f_{\text{M},s}(\mathbf{V}). \quad (5)$$

By using this form in  $\tilde{C}_s[\bar{F}_{\text{IM}}]$  and working out the  $\mathbf{V}$  integral (holding  $\mathbf{X}$  fixed) of  $\tilde{C}_s[\bar{F}_{\text{IM}}]$ , **show that you can efficiently recover Braginskii's result for cross-field diffusion of small density perturbations in a two-species plasma to lowest order in a gyroradius expansion.** Assume constant temperatures with  $T_e = T_i = T$ .

**Important notes and hints:**

- Most importantly, demonstrate that you understand which term(s) in the collision operator gives rise to the diffusion coefficient, and that the integral(s) you will have to do will lead to the proper parameter scaling. Only after you have done that should you worry about minus signs and the actual evaluation of any integrals.
- Do not attempt to do any sort of Chapman–Enskog solution of the kinetic equation. As stated above, directly evaluate the contribution to the gyrocenter density evolution equation of the collision operator acting on the assumed local Maxwellian.

## II.4: General Phenomena [10 points]

- (a) [5 points] White (visible) light incident on a piece of gold (density 19.3 g/cc, 197 AMU) is reflected. Why? Be quantitative.
- (b) [3 points] If the piece of gold is made thin enough, some light will leak through. Why and how thin? Be quantitative.
- (c) [2 points] Remarkably, the light that gets through is red-ish, not blue-ish. Explain why this is remarkable and why it happens.

## II.5: Waves and Instabilities [50 points]

A recent Physical Review Letters issue features an article<sup>1</sup> that describes the following effect: in a cold, collisionless, one-dimensional electron plasma with a warm beam, flattening of the beam distribution near the plasma resonance does *not* guarantee linear stability. Here you are asked to rederive this result, complement the original derivation, and explain the underlying basic physics.

(a) [5 points] First, consider plasma with a smooth low-density beam distribution  $F_0(v)$  without a plateau [Fig. 1(a)]. Express the dielectric function  $\bar{\epsilon}(\omega, k)$  as a functional of  $F_0$  and derive approximate equations for  $\omega_r \equiv \text{Re } \omega$  and  $\omega_i \equiv \text{Im } \omega$  [ $\propto F'(\omega_r/k)$ ].

**Hint:** Consider the following formula (you should know what it means) as given:

$$\chi_s = -\frac{\omega_{ps}^2}{k^2} \int \frac{f'_{0s}(v)}{v - \omega/k} dv. \quad (1)$$

(b) [30 points] Consider a beam distribution  $F(v)$  with a horizontal plateau of width  $2\Delta v$  centered around some velocity  $v_c$  [Fig. 1(b)]. The new dielectric function is  $\epsilon = \bar{\epsilon} + \delta\epsilon$ , where  $\delta\epsilon$  is due to  $\delta F(v) \equiv F(v) - F_0(v)$ . Sketch  $\delta F(v)$ , then sketch  $\delta F'(v)$ . Assuming that  $\Delta v$  is small enough such that  $F''_0$  is negligible across the plateau, calculate  $\delta\epsilon$  as a function of  $w \equiv (\omega - kv_c)/(k\Delta v)$ . Then derive the dispersion relation for  $w$ . (For simplicity, assume  $k > 0$ ,  $|w_r| < 1$ , and  $w_i \neq 0$ ; here  $w_r \equiv \text{Re } w$  and  $w_i \equiv \text{Im } w$ .)

**Hint:** Use the Taylor expansion of  $\bar{\epsilon}$  in  $w$ . The dispersion relation will have the form

$$\epsilon + \beta \left( \mu w + \ln(1 - w) - \ln(1 + w) + \frac{2w}{w^2 - 1} \right) = 0, \quad (2)$$

where  $\epsilon$ ,  $\beta$ , and  $\mu$  are real and independent of  $w$ . [Note that the imaginary unit,  $i$ , does not enter Eq. (2) explicitly.] The branch cut of  $\ln z$  is chosen as in Fig. 2.

(c) [15 points] Explore and sketch the solutions of Eq. (2) in the particular case  $\epsilon = 0$ :

1. Consider  $w \ll 1$ . Show that at  $\mu < 4$  one has  $\omega_r \equiv \text{Re } \omega = kv_c$ , and yet there is an instability. Why is this instability not suppressed despite  $F'(\omega_r/k) = 0$ ?
2. At  $\mu \ll 1$ , the imaginary roots grow large,  $w_1 = -w_2^* \gg 1$ . Find  $w_{1,2}$  in the limit  $w_{1,2} \rightarrow \pm i\infty$ . Show that one of them,  $w_1$ , is the standard rate of the bump-on-tail instability, the same as that found in part (a). Explain why this is anticipated.

(Problem continues on next page.)

<sup>1</sup> M. K. Lilley and R. M. Nyqvist, *Formation of Phase Space Holes and Clumps*, Phys. Rev. Lett. **112**, 155002 (2014).

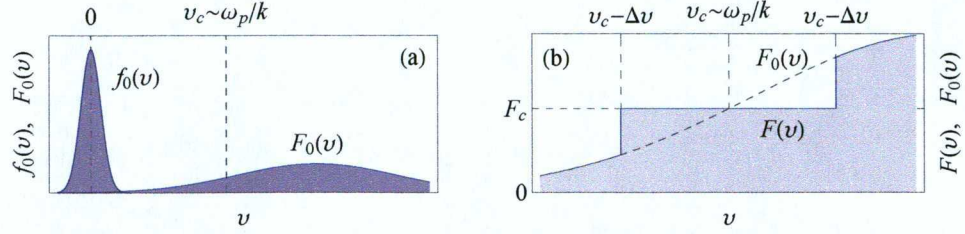


FIG. 1 (a) Distribution without a plateau: cold bulk,  $f_0$ ; warm beam,  $F_0$ . (b) Close-up of the beam distribution with a plateau,  $F$ . For reference,  $F_0$  is also shown (dashed).

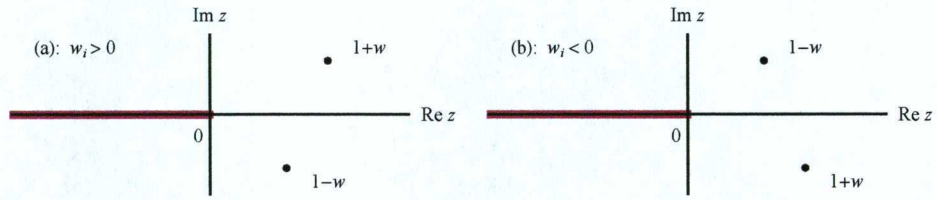


FIG. 2 Assumed branch cut (thick) for  $\ln z$  on the complex  $z$  plane. For your convenience, the points  $1+w$  and  $1-w$  are shown too, for  $w_i > 0$  (left) and  $w_i < 0$  (right). Also note the following:

$$\ln z = \ln |z| + i \arg z,$$

$$\ln(1 + \xi) = \xi - \xi^2/2 + \xi^3/3 - \dots \quad (\xi \ll 1).$$

## II.6: Computational Physics [20 points]

A colleague proposes to use the following algorithm to solve a system of equations of the form  $dy/dt = F(y)$  (where  $y$  and  $F$  could be vectors):

$$y_{n+1} = y_{n-1} + 2\Delta t F(y_n),$$

where  $y_n = y(t_n)$  and  $t_n = n\Delta t$ . He has shown that because this algorithm is centered in time, it ensures exact energy conservation when applied to the problem of particles gyrating in a magnetic field (if the time step is sufficiently small that it satisfies a Courant-type limit). He is correct in this. Now apply the above equation to a simple one-dimensional problem that includes drag:

$$\frac{dy}{dt} = -(\nu + i\Omega)y. \quad (1)$$

( $\nu$  is a drag rate and  $\Omega$  is a gyration frequency.) Show that this algorithm is always unstable for any non-zero drag rate  $\nu$ , no matter how small the time step is.

**Hint:** Show that there are two eigenmodes, and that one of them corresponds to an instability. The other root is a second-order-accurate approximation to the correct damped solution.