DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

May 11, 2015

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name and the question number on the title page of each booklet.
- When you do not have time to put answers into forms that satisfy you, indicate ***specifically*** how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem ___" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

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I.1: General Plasma Physics [40 points]

A planar diode in which the cathode is a copious emitter of electrons is limited in the amount of current it carries because of space-charge effects. The law is that the maximum current is proportional to the voltage, ϕ_0, to the 3/2 power. Assume two electrodes, one at $x = 0$ held at potential $-\phi_0$ and one at $x = L$ held at zero potential.

(a) [**6 points**] Sketch the plasma potential as a function of x.

(b) [**6 points**] Sketch the fluid velocity as a function of x.

(c) [**28 points**] Use Poisson's equation, together with an equation for divergence-free steady-state flow in one dimension (Hint: show that the current is constant) to derive the important "3/2" law for space-charge limited diodes. Explain the origin of any equations or boundary conditions that you use.

Hint: The equation $y_{xx} = y^{-1/2}$ may be approached by multiplying both sides by $y_x \equiv \partial y/\partial x$.

I.2: Waves and Instabilities [20 points]

(a) [10 points] Consider one-dimensional plasma with stationary homogeneous ions and an electron distribution $f_0(v)$ that is flat (constant) up to velocities that, in the context of this problem, can be considered infinite. Assume that, on top of this flat distribution, there is a hollow of depth $n_h/(2\bar{v})$ in some velocity interval $-\bar{v} < v < \bar{v}$, where $n_h > 0$ is some constant with units of density. Calculate the dielectric function of such plasma.

(b) [5 points] Calculate the dispersion relation of waves supported by such plasma and explore their stability. (Hint: Consider more than one type of boundary conditions.)

(c) [5 points] Calculate the energy density of these waves as a function of the wave amplitude and wave number. What can you say about the sign of this energy density?

I.3: Plasma Diagnostics [25 points]

(a) [15 points] Estimate the required energy for a ruby laser with a diameter of 1mm to measure electron density using incoherent Thomson scattering. Use the following conditions or parameters.

- Total Thomson scattering cross section $\sigma \sim 0.67 \times 10^{-24}\text{cm}^2$.
- Planck constant $h = 6.6 \times 10^{-34} J \cdot s$.
- Ruby laser wavelength $\lambda = 694.3\text{nm}$.
- Electron density $n_e = 10^{14}\text{cm}^{-3}$.
- Spatial resolution of $\Delta d = 1\text{cm}$.
- Detector area of 1cm^2 is located 100cm away from the scattering volume.
- Desired number of photons to arrive at the detector is more than 10^4.
- Laser is randomly polarized so scattering is isotropic.

(b) [5 points] What is the best direction to place the detector if the laser is instead polarized?

(c) [5 points] If using this best direction, how much reduction can we expect for the required laser energy?

I.4: MHD [30 points]

In astrophysical plasmas, flux tubes are a concept of great interest. T. Gold and F. Hoyle developed one of the first models of flux tubes in which magnetic field lines are twisted due to external forcing. Consider a Gold-Hoyle flux tube, which can be modeled as a straight cylindrical plasma in which equilibrium depends only on radius.

(a) **[12 points]** Assume that the axial and azimuthal magnetic fields are such that any magnetic field line rotates the same number of radians k per unit length. Assuming that the magnetic field is in a force-free equilibrium state and that the total magnetic field is B_0 on the axis, determine the magnetic field profile inside the flux tube. Sketch the field components as a function of radius.

(b) **[10 points]** Assume that the flux tube has a boundary at $r = a$. Obtain an expression for the total axial magnetic flux Φ contained in the cylinder as a function of B_0, k, and a. If there is gas of pressure P outside of the flux tube boundary, find a relation between P and B_0, k and a.

(c) **[8 points]** Describe how B_0 changes as we vary the twist k, holding Φ and P constant.

I.5: Kinetic Theory Quickie [15 points]

Consider the collisionless Vlasov equation for electrons interacting with an electrostatic potential from the Poisson equation (and no magnetic field). Take the ions as a stationary uniform neutralizing background. Show that energy is conserved. (You can use simple boundary conditions to simplify the boundary terms.)

I.6: Applied Mathematics [40 points]

The following fourth-order differential equation appears in fluid or plasma physics problems with non-uniform media.

$-\frac{\gamma}{t}$

$$\frac{1}{\Lambda}\frac{d^4y}{dx^4} + x\frac{d^2y}{dx^2} + \gamma y = 0. \tag{1}$$

The fourth-order differential operator represents the viscous or thermal correction, which becomes important only in a narrow region near $x \sim 0$ when Λ is large. The equation was studied in hydrodynamics by Orr (1907) and Sommerfeld (1908), and in plasma physics by Stix (1965) and Gorman (1966), but its mathematical structure was examined in details first by Wasow (1950).

(a) [6 points] Find the asymptotic behavior of the solutions for large x limit, using the eikonal form $y = e^{S(x)}$ and the expansion $S(x) = \sum S_n(x)$, but only up to the leading order in $S_n(x)$, i.e. $S_0(x)$. Assume that the solutions have irregular singular behaviors when $|x| \to \infty$.

(b) [6 points] Give the integral representations of the solutions using Fourier-Laplace Kernel, $y = \int_C e^{xt} f(t)dt$, with the conditions for the integral path.

(c) [8 points] Discuss all the possible integral paths in the complex t plane, when $\gamma < 0$.

Hint: Don't miss $t = 0$ which is non-analytic with essential singularity. Wasow discussed 7 different paths but obviously only 4 paths are independent to each other.

(d) [12 points] Find the saddle points and evaluate the leading asymptotic behavior of the solutions for large x limit, by the steepest-descent integration through each saddle. Use $\Lambda x^2 \gg 1$ to take only the leading term in x and to simplify the expressions. Can you recover the answers in (a)?

(e) [8 points] In plasma wave theory, the four leading behaviours represent fast or slow, and propagating or evanescent waves, with the time dependence $e^{-i\omega t}$. Consider one contour path that starts from $t = 0$ and ends at a point of infinity through a saddle point located at $\arg(t) = \pi/2$ (i.e. imaginary axis in the upper plane), when $\gamma < 0$. Discuss the solution behaviors for $x < 0$ and $x > 0$, and explain that your solutions can provide an analytic way to understand the mode conversion process between fast and slow waves.

I.7: Waves Quickie [10 points]

Consider a stationary O wave propagating in cold inhomogeneous plasma. Suppose that the wave frequency is given and that the electron density profile in the region of interest is

$$n(x) = \left(2 - x^2/\ell_0^2\right) n_0,$$

where n_0 is the critical density, and ℓ_0 is some characteristic scale. Estimate $\ln \mathcal{T}$, where $\mathcal{T}$ is the coefficient of the wave transmission through the region where the wave is evanescent.

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DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

May 12, 2015

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
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II.1: Waves and Instabilities [40 points]

Consider an electromagnetic wave propagating approximately parallel to static magnetic field $\vec{B}$ such that $B \approx (1 + z/\ell_0)B_0$. Here z is the coordinate along the field, ℓ_0 is some large constant, which can be positive or negative, and B_0 is the field strength at which the wave is in cyclotron resonance with some minority ions.

(a) [25 points] Assuming that the wave amplitude and frequency are given and that, in other respects, the plasma can be treated as cold, calculate the *total* wave power (per unit cross section of the wave beam) that is absorbed in plasma through cyclotron heating of the minority ions.

Note: If you do not remember formulas that you consider necessary to solve this problem, explain how you would approach it if you had those formulas available. Note, however, that the expression for the *hot* plasma dielectric tensor is not needed here.

(b) [6 points] Propose an analytical estimate for the characteristic length of the absorption region, a_{abs}.

(c) [9 points] Assuming parameters typical for ion cyclotron heating in tokamaks, estimate a_{abs} numerically.

$1 - 1 - \frac{z}{\ell_c}$

$\frac{c^2 k_\parallel^2}{\omega^2} = \frac{\omega_p^2}{(1-\frac{\Omega}{\omega})}$

$\sqrt{\frac{z}{\ell_c}}$

$k^2 \quad \frac{\omega_p}{c\sqrt{1-\frac{\Omega}{\omega}}}$

II.2: MHD Quickie [15 points]

You do not need to write down any equations to answer the following questions. A few sentences will suffice. (If you feel that it is quicker to write down a few relevant equations, that is also fine.)

(a) [**3 points**] Why does confinement of a plasma by a toroidal, axisymmetric magnetic field require a toroidal plasma current?

(b) [**5 points**] What is rotational transform, and why it is relevant to the previous question?

(c) [**7 points**] How does a stellarator produce rotational transform?

II.3: Irreversible Processes [50 points]

In the particle-in-cell method of solving the collisionless gyrokinetic equation, N discrete "marker" particles are used to sample phase space. Because N is finite, there is a fluctuation noise associated with the collection of markers, and that noise leads to a spurious diffusion that is not present in the original physics problem. When one later goes on to study turbulent diffusion due to drift waves, the spurious diffusion can be problematical because it reduces the signal-to-noise ratio.

The basic question is as follows:

> ***Assuming that the gyrokinetic marker system is weakly coupled, find a formula for the cross-field test-gyrocenter diffusion coefficient $D_\perp$ as a function of N.***

Make the following assumptions:

- The gyrocenters are in thermal equilibrium [described by the Maxwellian distribution function $F_M(\boldsymbol{v})$]. That is, there are no profile gradients that would drive turbulence. This means that any diffusion is solely a result of the discreteness in the marker distribution.
- The magnetic field is constant.
- The electric field $\boldsymbol{E}$ can be calculated in the electrostatic approximation.
- The electrons obey adiabatic response. This means that the only discreteness is due to N ions.
- Do not assume that $T_i = 0$. Retain finite-Larmor-radius (FLR) corrections to both the kinetic equation and the gyrokinetic Poisson equation. See the hints below for the form of that latter equation.

(a) [20 points] ***Provide in words a very detailed outline of how you would proceed to solve this problem. For each step in your outline, also describe the relevant physical or mathematical justification.***

You are being given a substantial amount of time for this part. Take this opportunity to prove that you have a broad and deep understanding of the relevant kinetic theory of irreversible processes. Don't be too succinct. Don't just say something like ". . . then use Theorem X." If you mention a theorem or formula, explain what its content is and where it came from.

You could, of course, write an entire essay on irreversible processes. Don't write more than 20 minutes worth of material. And what you write must be relevant to the gyrokinetics problem stated above.

(b) [30 points] ***Proceed toward the goal of finding a general formula for $D_\perp$.*** Do as many mathematical calculations as you can in the time allowed. If you have to make a choice, focus on the calculation of the fluctuation level.

(Problem continues on next page.)

Hints:

$\nabla_\perp^2 \, b$ $\quad$ $\nabla \cdot \int f \; E \times B$

- Gyrocenter physics includes
  - parallel streaming and acceleration along the magnetic field;
  - perpendicular $\boldsymbol{E} \times \boldsymbol{B}$ motion.
- The effective potential $\varphi \stackrel{\text{def}}{=} e\phi/T_e$ felt by a gyrocenter is in $\boldsymbol{k}$ space $J_0\varphi_{\boldsymbol{k}}$, where $J_0 \equiv J_0(k_\perp v_\perp/\omega_c)$. The Bessel function J_0 is only different from 1 for the ions.
- In the presence of FLR effects, the gyrokinetic Poisson equation is

$$-\tau(1-\Gamma_{\boldsymbol{k}})\widetilde{\varphi}_{\boldsymbol{k}} = \frac{1}{\overline{n}_i}\int d\boldsymbol{v}\; J_0\widetilde{N}_{\boldsymbol{k},i}(\boldsymbol{v}) - \frac{1}{\overline{n}_e}\int d\boldsymbol{v}\; \widetilde{N}_{\boldsymbol{k},e}(\boldsymbol{v}), \tag{1}$$

where $\widetilde{N}$ is the microdensity of gyrocenters,

$$\tau \stackrel{\text{def}}{=} ZT_e/T_i, \tag{2}$$

$$\Gamma_{\boldsymbol{k}} \stackrel{\text{def}}{=} \int d\boldsymbol{v}_\perp\; J_0^2 F_{M,i}(\boldsymbol{v}_\perp) = I_0(b)e^{-b}, \tag{3}$$

I_0 is a modified Bessel function, and

$$b \stackrel{\text{def}}{=} k_\perp^2\rho_i^2. \tag{4}$$

Note that as $k_\perp\rho_i \to 0$, one has

$$\tau(1-\Gamma) \to k_\perp^2\rho_s^2, \tag{5}$$

where

$$\rho_s \stackrel{\text{def}}{=} c_s/\omega_{ci}. \tag{6}$$

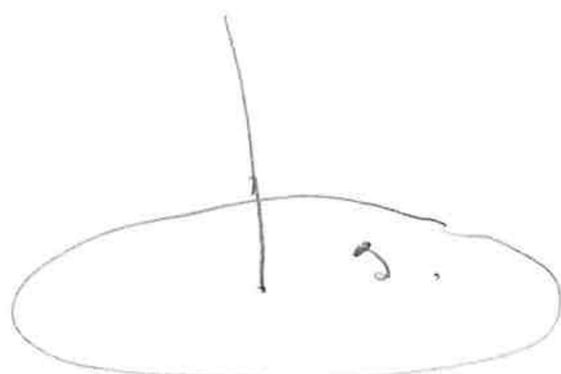
This reproduces the cold-ion limit that may be more familiar to you. But there is no need to take that limit in this problem.

- The density of a test gyrocenter contains a Bessel function.

II.4: Elementary Physics Quickie [10 points]

Consider a simple circular current-carrying wire loop of radius a, lying in the $x - y$ plane and with its major axis along the z-axis. Describe the motion of a charged particle in two cases, with the following characteristics:

- The particle is initially located at $z = +0.01$ and have $v_z = 0$.
- The particle has a sufficiently low energy such that its Larmour radius is $0.1a$.
- One case is initially at $r = 1.5a$, and the other case at $r = 0.1a$ in which gyro-orbit encircles the major axis.



II.5: Experimental Methods [35 points]

The current-voltage characteristic shown in Figure 1 was obtained from a single-tipped Langmuir probe, in an unmagnetized laboratory plasma.

(a) [**3 points**] Identify the ion saturation, electron saturation, and transition regions in the probe trace.

(b) [**3 points**] Which bias conditions define these regions?

(c) [**5 points**] Indicate the approximate values for the floating potential and space potential (on the trace itself, and numerically).

(d) [**5 points**] If the ion species is singly-ionized argon, use the results of (c) to estimate the electron temperature. Alternatively, use the appropriate region of the probe trace to experimentally determine the electron temperature.

(e) [**7 points**] If the probe tip (the conductor immersed in the plasma) is 1 mm in diameter and 5 mm long, what is the approximate value of the plasma density (again, assume singly charged argon for the ionic species)? Assume $T_e \gg T_i$. Are sheath expansion effects likely to affect the accuracy of the density estimate? Explain.

(f) [**6 points**] Discuss any possible changes to the values you obtained in parts (c), (d), and (e) which might result if the plasma is strongly magnetized.

(g) [**6 points**] Other than density, can any information regarding the ion population be obtained from the probe I-V trace?

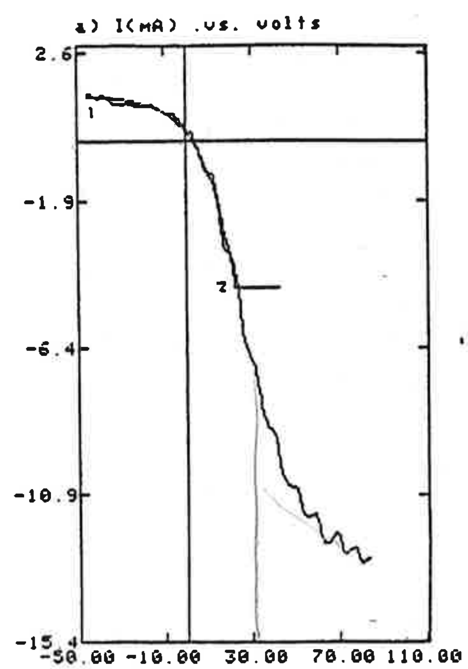


FIG. 1 The current (mA, Y-axis) vs. voltage (V, X-axis) charateristics measured by a Langmuir probe in laboratory.

II.6: Neoclassical Theory [30 points]

(a) [15 points] (a) Use the conservation of canonical angular momentum

$$P_\zeta = mRv_\zeta + \frac{e}{c}\psi \tag{1}$$

to estimate the width of the banana orbit as a multiple of the particle's Larmor radius.

(b) [15 points] Describe and give (without derivation) the effective collision frequency in the "banana regime", fraction of trapped particles, and "random walk" estimate for the cross-field particle diffusivity D_{banana}.

Note: You can use a simpler estimate of the banana width here if you are unable to carry out part (a).