DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

May 16, 2016

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name and the question number on the title page of each booklet.
- When you do not have time to put answers into forms that satisfy you, indicate ***specifically*** how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem ___" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

Contents

Page

I.1: General Plasma Physics [30 points]

Consider N non-relativistic charged particles interacting through a self-consistent electromagnetic field in a flat space-time. Here, the total number of charged particle N is a fixed number, e.g., $N = 3$ or $N = 10^{21}$. The dynamics of the j-th particle with charge e_j and mass m_j ($j = 1, ..., N$) is given by a curve (trajectory) in the space-time $\boldsymbol{X}_j(t)$. The electromagnetic field is given by $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$.

(a) [5 points] Write down the complete set of equations that governs the dynamics of $\boldsymbol{X}_j(t)$, $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$. Explain why the system should be called Newton-Maxwell system. You will need the following function,

$$\delta_2 \equiv \delta(\boldsymbol{X}_j - \boldsymbol{x})\,. \tag{1}$$

(b) [5 points] The system admits the following local energy conservation law,

$$\frac{\partial}{\partial t}\left[\frac{\boldsymbol{E}^2 + \boldsymbol{B}^2}{8\pi} + \sum_j \frac{m_j \dot{\boldsymbol{X}}_j^2}{2}\delta_2\right] + \nabla \cdot \left[\frac{c\boldsymbol{E} \times \boldsymbol{B}}{4\pi} + \sum_j \frac{m_j \dot{\boldsymbol{X}}_j^2}{2}\dot{\boldsymbol{X}}_j \delta_2\right] = 0. \tag{2}$$

Write down the local momentum conservation law.

(c) [5 points] It is generally believed in physics that space-time symmetry is the reason for the energy-momentum conservation. What is the symmetry responsible for the energy-momentum conservation for the Newton-Maxwell system?

(d) [15 points] Prove that the governing equations of the Newton-Maxwell system that you wrote down indeed admits the local energy conservation law, i.e., Eq. (2).

I.2: Plasma Diagnostics [25 points]

(a) [12 points] Many materials (especially insulators, e.g. boron, but also conductors in certain energy ranges) have high secondary electron emission coefficients, for electron impact. What would be the effect of a high secondary electron emission coefficient on the floating potential of an electric probe, if the probe tip were made of such a conducting material?

(b) [8 points] Now specifically consider the effects of secondary electron emission coefficients $\delta_e = 0.5$ and $\delta_e = 0.9$. Assume that the probe is immersed in a fully ionized hydrogen plasma.

(c) [5 points] Could the use of probe tips with varying secondary electron emission coefficients be used in principle to estimate the plasma space potential?

I.3: Waves and Instabilities [45 points]

Consider a collisionless relativistic beam of charged particles in a storage ring (a confinement system, typically made of magnets, that is intended to make particles go around approximately closed orbits). For clarity, assume particle orbits to be one-dimensional and circular, so they can be described in terms of the azimuthal angle θ and the corresponding canonical momentum J. Absent collective interactions, the particle motion is governed by some θ-independent Hamiltonian $H_0(J)$ determined by the confining system; then J is conserved and particles circulate with fixed angular frequencies $\Omega(J) = H_0'(J)$. Collective interactions alter this picture by introducing a perturbation potential $\tilde{\phi}(t, \theta)$, so the Hamiltonian becomes $H(t, \theta, J) = H_0(J) + e\tilde{\phi}(t, \theta)$, where e is the particle charge. Below, you are asked to explore the beam stability with respect to such collective interactions without specifying H_0.

(a) [**7 points**] Assuming that $\tilde{\phi}$ is small, consider the dynamics of the particle distribution $f(t, \theta, J)$ around an unperturbed equilibrium f_0. Show that $f_0 = f_0(J)$ and argue that the linearized Vlasov equation for $\tilde{f}(t, \theta, J) \overset{\text{def}}{=} f - f_0$ is

$$\partial_t \tilde{f} + \Omega(J)\, \partial_\theta \tilde{f} = -e\tilde{E}Rf_0'(J), \tag{1}$$

where $\tilde{E}$ is the electric field corresponding to $\tilde{\phi}$, and R is the storage ring radius.

(b) [**13 points**] Using Eq. (1), derive the general dispersion relation for linear eigenmodes in a beam. (For simplicity, adopt $\partial_t = -i\omega$ and assume that $\tilde{E}$ is approximately one-dimensional due to the shielding provided by the storage ring walls.) If your answer involves an integral, make sure to specify the path along which the integral is taken. In particular, assuming $f_0(J)$ is smooth, how would you take the integral for real ω?

(c) [**15 points**] Consider a cold beam centered in the momentum space around some J_0. Derive the mode frequencies explicitly and find the condition on $\Omega(J)$ under which a beam is unstable. In particular, if a relativistic beam is confined by a uniform magnetic field transverse to the plane of the storage ring, will the beam be stable or unstable?

(d) [**10 points**] If you solved part (c) correctly, you may notice that the obtained dispersion relation is somewhat similar to that of electron Langmuir oscillations (ELO) in cold boundless plasma. Then, would you say that ELO in cold boundless plasma can exhibit a similar instability? In either case, try to explain qualitatively the mechanism of the instability found in part (c).

I.4: Experimental Quickie [15 points]

A D-T fusion reaction produces a 3.5 MeV alpha particle. A tokamak must confine alphas if it is to approach ignition. Estimate the drift orbit width for a trapped alpha particle in a tokamak, mirroring at the maximum $|\boldsymbol{B}|$ (i.e. the fattest banana orbit), in a tokamak, for $B = 5T$, $R/r = 4$, $q = 2$.

I.5: Kinetic Theory Quickie [15 points]

Intuitively one usually visualizes correlation functions as decaying to zero over some autocorrelation time τ_{ac} that measures the area under the curve, as in Fig. 1. But in an equilibrium, unmagnetized, weakly coupled, classical, discrete plasma, the correlation function for the acceleration ($\sim$ force) on a test particle, $C_{aa}(\tau) \stackrel{\mathrm{def}}{=} \langle \dot{v}'(\tau)\dot{v}'(0)\rangle$ (prime denotes fluctuation) has a long negative tail (Fig. 2) such that the total area under the curve is zero.

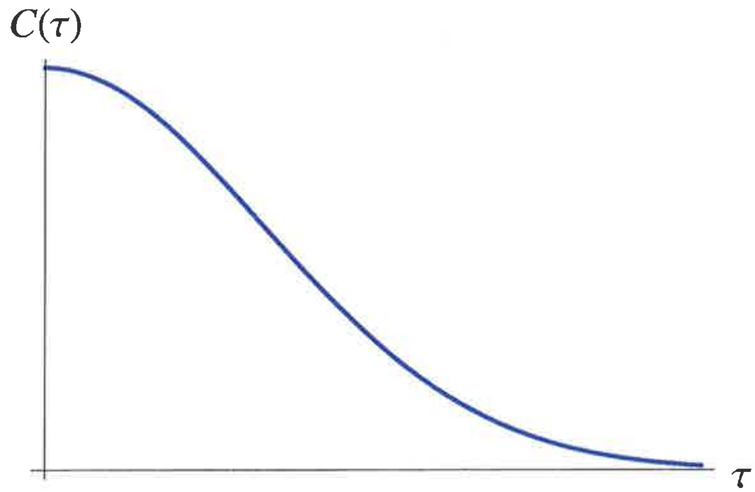

FIG. 1 The right-hand side of a generic two-time correlation function.

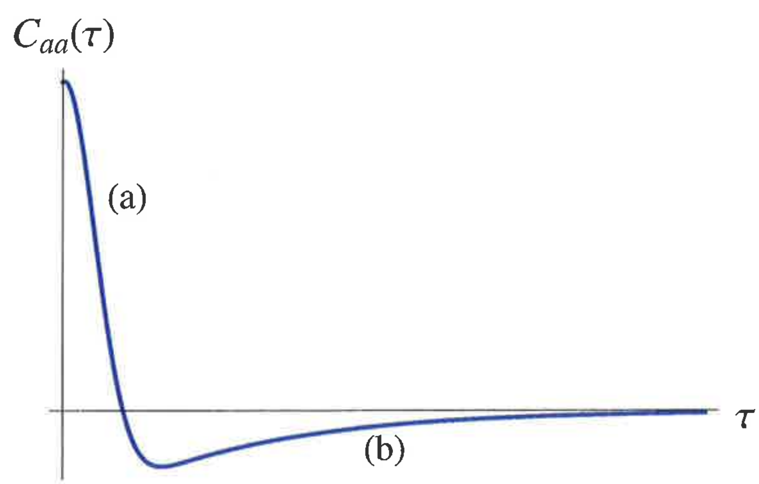

FIG. 2 The right-hand side of the two-time correlation function $C_{aa}(\tau)$ for the acceleration or force on a test particle.

(a) [7 points] Explain the physics behind the curve shown in Fig. 2, including an identification of the time scales associated with parts(a) and (b) of the curve as well as a clear statement of why physically the negative tail exists.

(b) [8 points] Starting with the simplest plausible model for the velocity correlation function of a test particle (use the exact result if you know it), differentiate it appropriately to find a form for $C_{aa}(\tau)$ that looks like Fig. 2.

I.6: MHD [40 points]

In this problem, you will be asked to derive interchange stability condition of ideal MHD plasmas confined by gravity using energy principle in both low-β and high-β limits. When $\beta \gg 1$, the plasma equilibrium is given by

$$\rho \mathbf{g} = \nabla P \tag{1}$$

where ρ is plasma density, P is plasma pressure and $\mathbf{g}$ is gravity vector. The energy integral in this case is given by

$$\delta W = \frac{1}{2} \int dV \left[\gamma P (\nabla \cdot \xi)^2 + (\xi \cdot \nabla P) \nabla \cdot \xi + (\xi \cdot \mathbf{g}) \nabla \cdot (\rho \xi)\right] \tag{2}$$

where ξ is displacement vector.

(a) [12 points] What value should plasma compressibility, $\nabla \cdot \xi$, take in order to minimize δW?

(b) [8 points] For stability, what sign should $\nabla(P/\rho^\gamma)$ take with respect to $\mathbf{g}$ (Schwarzschild Criterion)?

(c) [5 points] For plasmas with $\beta \ll 1$, the equilibrium becomes

$$\rho \mathbf{g} = \nabla P_B \tag{3}$$

where $P_B = B^2/(2\mu_0)$. In this case, we can treat magnetic field as a massless fluid with a finite pressure of P_B. What value should γ take for such a fluid? [Hint: γ is defined so that $PV^\gamma = const.$ For magnetic field, $V \propto 1/B$ is volume per flux.]

(d) [5 points] What is the revised stability condition for the equilibrium given by Eq.(3)?

(e) [10 points] You might notice that, when we move from high-β plasmas to low-β plasmas, we use the same δW given by Eq.(2) where the perturbed magnetic field is ignored while the background magnetic field is strong. Is this assumption justified? [Hint: the perturbed magnetic field is given by $\mathbf{Q} \equiv \nabla \times (\xi \times \mathbf{B})$ where no line-bending is allowed in interchange modes. You can use vector identity: $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$.]

I.7: Waves Quickie [10 points]

Suppose a stationary plane electromagnetic wave with electric field amplitude $\boldsymbol{E}_0$, frequency ω_0, and wave vector $\boldsymbol{k}_0$ is incident from vacuum normally on a stationary plasma and transforms without reflection into some linear dissipationless electrostatic wave. Assume that the transformation is adequately described by geometrical optics. Both the wave vector $\boldsymbol{k}$ and the group velocity $\boldsymbol{v}_g$ of the electrostatic wave are everywhere parallel to $\boldsymbol{k}_0$. Away from the boundary, the plasma is homogeneous and has some general dielectric tensor $\hat{\boldsymbol{\epsilon}}(\omega, \boldsymbol{k})$.

(a) [**5 points**] Without assuming a specific $\hat{\boldsymbol{\epsilon}}$ or specific waves, write down the general <u>electrostatic</u> dispersion relation for the wave in the plasma. Express $\boldsymbol{v}_g$ of the electrostatic wave through $\hat{\boldsymbol{\epsilon}}(\omega, \boldsymbol{k})$. Express the wave frequency in plasma, ω, through ω_0.

(b) [**5 points**] Express the amplitude of the electrostatic-wave potential $|\phi|$ through $\boldsymbol{E}_0$, ω_0, and $\boldsymbol{k}$. Simplify your answer using results obtained in part (a).

DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

May 17, 2016

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
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Contents

Page

II.1: General Plasma Physics [15 points]

Consider a magneto-electric particle trap in the region $-L < z < L$. To accomplish this trap, suppose a magnetic field in the z direction such that

$$B = \begin{cases} B_0 \left(1 + (R-1)\frac{z^2}{L^2}\right), & \text{if } -L < z < 0; \\ B_0, & \text{if } z \geq 0. \end{cases}$$

Suppose also an electric potential

$$\phi = \begin{cases} 0, & \text{if } z < 0; \\ \phi_0 \left(\frac{z^2}{L^2}\right), & \text{if } 0 \leq z < L; \\ \phi_0, & \text{if } z > L. \end{cases}$$

(a) [**6 points**] Describe how ions might be trapped in this configuration of magnetic and electric fields. Would electrons also be trapped in the same fields? Sketch the trapping condition in midplane perpendicular and parallel energy space ($W_{\perp 0}$-$W_{\| 0}$ space).

(b) [**4 points**] Suppose that the ions undergo collisions with a buffer gas. If trapped ions of charge state q were scattered in pitch-angle but not in energy through collisions, from what end of the device would they leave? Please explain very briefly (in one sentence). If trapped ions of charge state q were slowed down in energy through collisions without undergoing pitch-angle scattering, from what end of the device would they leave? Please explain very briefly (in one sentence).

(c) [**5 points**] Suppose that two species of ions, with different charge to mass ratios, are initially trapped. Suppose that these ions they undergo infrequent collisions in both pitch angle and energy with a buffer gas. Suppose that that an rf source is available to inject waves into the device and that it can be tuned to the ion cyclotron resonance. Briefly explain how might you use this device to separate one ion species from the other?

II.2: Irrevesible Processes [40 points]

Consider a statistically uniform plasma in a periodic domain of volume V.

(a) [10 points] Show how the 2-point correlation function for any quantity $\tilde{A}(x)$ (with mean value $\langle \tilde{A} \rangle = 0$) can be written in terms of the power spectrum, $\langle |\tilde{A}_k|^2 \rangle$, where angle brackets indicate ensemble averaging and

$$\tilde{A}(x) = \sum_k e^{i\vec{k}\cdot\vec{x}} \tilde{A}_k$$

(b) [20 points] In the electrostatic limit, the electric field spectrum in such a plasma in thermal equilibrium at temperature T can be written as (in cgs):

$$\frac{\langle |\tilde{\vec{E}}_k|^2 \rangle}{8\pi} = \frac{1}{V}\frac{T}{2}\frac{1}{(1 + k^2\lambda_D^2)}$$

From the corresponding $\langle |\phi_k|^2 \rangle$ spectrum, derive the rms electric potential fluctuation amplitude, i.e., $\Phi_{rms}^2 \equiv \langle \Phi^2(x) \rangle$. Express $e\Phi_{rms}/T$ in terms of common plasma parameters. (You should simplify your final answer by approximating k-summations with integrals.)

(c) [10 points] Briefly give a physical interpretation (or back-of-the-envelope estimate) of the result.

II.3: Elementary Physics Quickie [15 points]

(a) [5 points] Calculate the radial pressure profile in an infinitely long, cylindrical (radius a), plasma column carrying a uniform axial current of density j_z inside $r = a$.

(b) [5 points] If an identical Z-pinch is placed paraxially, but not coaxially, near the first, is it attracted or repelled?

(c) [5 points] To what type of instabilities is this configuration susceptible?

II.4: Applied Math [35 points]

Consider the hyper-Airy differential equation

$$\frac{d^4y}{dx^4} = xy,$$

which appears as a special case in the applications of fractional differential equation and can model some of diffusion processes in complex media.

(a) [**5 points**] Find the leading asymptotic behaviors of the solutions for large positive x, using the eikonal form $y = e^{S(x)}$. Assume that the solutions have irregular singular behaviors when $x \to +\infty$.

(b) [**5 points**] Find the next order asymptotic behaviors of the solutions for large positive x, by expanding the eikonal function, as $S(x) = S_0(x) + S_1(x)$ for example.

(c) [**5 points**] Give the integral representations of the solutions using Fourier-Laplace Kernel, $y = \int_C e^{xt} f(t)dt$, with the conditions for the integral path.

(d) [**5 points**] Discuss all the possible integral paths in the complex t-plane.

(e) [**8 points**] Recover the answers in (a) and (b), by finding the saddle points and evaluating the integral by the method of steepest descent.

(f) [**7 points**] Consider the boundary condition $y(0) = 1$ and $y(+\infty) = 0$. Is there a unique solution? Give your thoughts briefly and explaine how you would analytically find the solution if exists, and if possible, find the solution with an integral representation.

II.5: MHD Quickie [15 points]

We consider a magnetic field in slab geometry, with cooridinates (x, y, z), periodic boundary considitions in y, and with the field indpendent of z, $\boldsymbol{B} = \boldsymbol{B}(x, y)$. We are given a zeroth order magnetic field

$$\boldsymbol{B}_0 = \boldsymbol{\nabla}\psi_0(x) \times \hat{\boldsymbol{z}} + B_z \hat{\boldsymbol{z}},$$

where $\hat{\boldsymbol{z}}$ is a unit vector in the z direction, ψ_0 is a function only of the x coordinate, with $d\psi_0/dx = 0$ at $x = 0$, and B_z is a constant. We add a small perturbing field $\boldsymbol{B}_1$, such that $\hat{\boldsymbol{z}} \cdot \boldsymbol{B}_1 = 0$, and $\hat{\boldsymbol{x}} \cdot \boldsymbol{B}_1 = \epsilon \sin(ky)$ at $x = 0$. Find a function $\psi(x, y)$ such that $\boldsymbol{B} \cdot \boldsymbol{\nabla}\psi = 0$. Sketch the shape of the surfaces of constant ψ.

II.6: Neoclassical Theory [40 points]

In an axisymmetric toroidal plasma, the conservation of canonical angular momentum can be used to estimate key neoclassical transport properties such as the inward particle pinch velocity (Ware Pinch) and the banana excursion of trapped particles in the long-mean-free-path banana regime. The conservation of canonical angular momentum, P_ζ, can be expressed in terms of the poloidal flux function, ψ, as follows:

$$P_\zeta = mRv_\zeta + \frac{e}{c}\psi.$$

(a) [**10 points**] Working with the expression given for P_ζ and taking $v_\zeta \approx v_\parallel$, show that trapped-particle orbit averaging simply gives: $\overline{\partial\psi/\partial t} = -\overline{\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi}$.

(b) [**10 points**] Since the Ware Pinch is just the orbit-averaged trapped-particle radial velocity, obtain an estimate for this quantity by using Faradays Law together with the result from part (a).

(c) [**5 points**] Using $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, express P_ζ in terms of B_θ.

(d) [**5 points**] Expand around r_0, the mean radius of a trapped-particle orbit, to express the result from part (c) in terms of the trapped-particle radial excursion, $\Lambda \equiv r - r_0$.

(e) [**10 points**] Taking $v_\zeta \approx v_\parallel \approx \epsilon^{1/2}v_{th}$ (with $\epsilon = r/R_0$) and $B_\theta \approx B_p$ (poloidal magnetic field), obtain an estimate for Λ in terms of the gyroradius.

II.7: Experimental Methods [20 points]

Plasmas may be cold, $0.1 - 10eV$, warm, *ca.* $10 - 100eV$, or hot, well above $100eV$. Name one (practical) diagnostic method that is used to measure the electron temperature for each range, sketch what the "raw" data would look like, and indicate on the sketch how the electron temperature is deduced. (You have freedom in this problem to choose plasma densities, ion species, and plasma sizes suitable for your methods.)