DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

May 15, 2017

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name and the question number on the title page of each booklet.
- When you do not have time to put answers into forms that satisfy you, indicate ***specifically*** how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem ___" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

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I.1: General Plasma Physics [40 points]

(a) [**15 points**] Consider a relatively cold plasma consisting of equal densities of electrons and positrons. Beginning with the cold plasma fluid equations in one dimension, derive the cold collisionless plasma dispersion relation $\omega^2 = \omega_*^2$, where ω_* is the oscillation frequency of the plasma oscillations. Explain all your approximations and limits of validity of your results.

Hint: *you may find it convenient to consider separately the difference density, $n_i - n_e$, and the sum density $n_i + n_e$, where n_i is the positron density and n_e is the electron density.*

(b) [**15 points**] Suppose the initial conditions for the electron-positron plasma are given by: $n_e(x, t = 0) = n_0 + g(x)$; $n_i(x, t = 0) = n_0$; $v_e(x, t = 0) = 0$; and $v_i(x, t = 0) = 0$, where $g(x)$ is small compared to n_0 and has zero area (when integrated over x). The electron and positron fluid velocities, given respectively by $v_e(x, t = 0)$ and $v_i(x, t = 0)$, initially vanish everywhere. Using these initial conditions, calculate the quantities: $n_e(x, t)$, $n_i(x, t)$, $v_e(x, t)$, and $v_i(x, t)$.

(c) [**5 points**] For an electron-positron plasma with temperature T, what condition on $g(x)$ insures that the cold plasma approximation is valid?

(d) [**5 points**] Suppose that a beam of electrons with velocity v_0 passes through a slab of electron-positron plasma with initial conditions as given. What condition on $g(x)$ insures that the electron beam will exactly conserve its phase-space density as it traverses the plasma slab?

I.2: General Plasma Physics [20 points]

Krook collision operator for one species is

$$\left(\frac{\partial f}{\partial t}\right)_{collision} = -\mu\left[f - f_M\left(\boldsymbol{v} - \langle\boldsymbol{v}\rangle\right)\right],$$

$$\langle\boldsymbol{v}\rangle \equiv \frac{\int f(\boldsymbol{v})\boldsymbol{v}d^3\boldsymbol{v}}{\int f(\boldsymbol{v})d^3\boldsymbol{v}},$$

$$f_M\left(\boldsymbol{v} - \langle\boldsymbol{v}\rangle\right) \equiv \frac{n}{(2\pi T/m)^{3/2}}\exp\left[-\frac{\left(\boldsymbol{v} - \langle\boldsymbol{v}\rangle\right)^2}{2T/m}\right],$$

where μ is a constant frequency and m is the mass of the species. Show that for properly chosen T and n, the collision operator conserves number density, momentum, and energy.

I.3: Experimental Methods [40 points]

(a) [10 points] What magnetic diagnostics are needed to determine the steady-state global energy confinement time τ_E for an Ohmically heated, circular-cross-section, large-aspect-ratio tokamak plasma?

(b) [30 points] Write an expression for the energy confinement time τ_E , using the quantities measured with the diagnostic set you propose. Assume that the plasma β is small, so that any plasma-induced deviation from the vacuum magnetic field is small.

I.4: MHD [45 points]

In plasma equilibria, cross-field particle drifts usually produce local charge accumulation and a corresponding electrostatic field. In the MHD equations, the effect of the cross-field drift on charge accumulation comes in through $\boldsymbol{j}_\perp$, the component of the current perpendicular to the magnetic field.

(a) [**5 points**] Using the MHD force balance equation, derive an expression for $\boldsymbol{j}_\perp$ in terms of the pressure gradient and magnetic field.

We will look at the effect of $\boldsymbol{j}_\perp$ on charge accumulation and on the electric field in the context of a relatively simple magnetic field,

$$\boldsymbol{B} = I\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi, \tag{1}$$

where $\psi = B_\theta r^2/2$, $I = I_0 - I_1\psi$, and where I_0, I_1 and B_θ are constants. Work in an orthogonal coordinate system (r, B_θ, B_ϕ), where r is the minor radius, θ is the conventional geometric poloidal angle, and ϕ is the conventional geometric toroidal angle. R is the major radius, with $R = R_0$ at the coordinate axis. Assume large aspect ratio, $r/R_0 \ll 1$. With this approximation, it will be convenient to write $R = R_0 + r\cos\theta$. Also assume that the magnitude of the toroidal field produced by the plasma current is small relative to that produced by the external magnetic field coils, $|I_0| \gg |I_1\psi|$, and that the toroidal field is large compared to the poloidal field, $|I_0| \gg |rB_\theta|$. We will further assume that we are given the pressure profile as $p(\psi)$ and that p is sufficiently small that the modification of $\boldsymbol{B}$ produced by the pressure driven current can be neglected.

Hint: *You will save time and effort in the following calculations if, early in your calculation, you discard terms in your expressions that are a factor $|rB_\theta/I_0|$ or $|I_1\psi/I_0|$ smaller than other terms.*

(b) [**10 points**] Evaluate the expression for $\boldsymbol{j}_\perp$ that you obtained in problem (a), above, to first order in r/R_0 and to lowest order in $I_1\psi/I_0$ and rB_θ/I_0, using the given field and the pressure profile in the form given above. (You can use $\boldsymbol{\nabla}p = p'(\psi)\boldsymbol{\nabla}\psi$.)

(c) [**10 points**] Calculate $\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp$ to lowest order in r/R_0, rB_θ/I_0, and $I_1\psi/I_0$. (There is a subtlety here because of the fact that both angular coordinates have a finite radius of curvature. The calculation can be done properly by taking advantage of the fact that $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}r\times\boldsymbol{\nabla}\theta) = \boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}\times(r\boldsymbol{\nabla}\theta)] = 0$ and $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}r) = 0$. If you don't see how to make use of these identities, then just do the calculation as if you were working in a cylindrical coordinate system with $\hat{\boldsymbol{z}} = \hat{\phi}$ and proceed from there.)

The divergence of $\boldsymbol{j}_\perp$ produces a local charge accumulation, which in turn produces an electrostatic field.

(d) [**5 points**] The MHD equations assume quasi-neutrality. Is it valid to use the (resistive) MHD equations in this context, where there is a charge accumulation and an associated electrostatic field? Why or why not?

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The electrostatic field produced by the charge accumulation drives a current along the field lines, called the *Pfirsch-Schlüter* current.

(e) [**10 points**] Using quasi-neutrality, solve for $\boldsymbol{j}_{\|} = j_{\|}\boldsymbol{B}/B$ to lowest order in r/R_0, rB_θ/I_0 and $I_1\psi/I_0$.

Hint: *You may find the following identity useful:* $\boldsymbol{\nabla}\cdot(j_{\|}\boldsymbol{B}/B) = \boldsymbol{B}\cdot\boldsymbol{\nabla}(j_{\|}/B)$.

(f) [**5 points**] Using the resistive MHD equations, solve for the electrostatic field along the magnetic field lines produced by the charge accumulation, assuming a uniform resistivity η.

I.5: Neoclassical Transport [35 points]

In an axisymmetric toroidal plasma, the conservation of canonical angular momentum can be used to estimate the *banana* excursion of trapped particles in the long-mean-free-path banana regime. The conservation of canonical angular momentum, P_ζ, can be expressed in terms of the poloidal flux function, ψ, as follows:

$$P_\zeta = mRv_\zeta + \frac{e}{c}\psi. \tag{1}$$

(a) **[10 points]** Show that $\partial\psi/\partial t = -\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi$ by assuming $v_\zeta = v_\parallel$.

(b) **[10 points]** Estimate the trapped-particle radial velocity by using Faraday's Law together with the result from part (a).

(c) **[5 points]** Using $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, now express P_ζ in terms of B_θ .

(d) **[5 points]** Expand around r_0, the mean radius of a trapped-particle orbit, to express result from part (c) in terms of the trapped-particle radial excursion: $\Lambda = r - r_0$.

(e) **[5 points]** Taking $v_\zeta \sim v_\parallel \sim \epsilon^{1/2}$ (with $\epsilon \equiv r/R_0$) and $B_\theta \sim B_p$ (poloidal B field), obtain an estimate for Λ in terms of the gyroradius.

DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

May 16, 2017

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
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II.1: Waves and Instabilities [50 points]

In this problem, you are asked to explore the effect of intense electromagnetic (EM) radiation on Langmuir waves in nonmagnetized homogeneous fluid plasma with immobile ions.

(a) [**15 points**] Suppose EM electric field $\mathbf{E}_{\text{em}} = \text{Re}(\mathbf{E}_c e^{i\theta})$, where $\omega \equiv -\partial_t \theta$, $\mathbf{k} \equiv \nabla\theta$, and $\mathbf{E}_c$ are slow compared to θ. Assuming ω is much larger than the plasma frequency ω_p, argue that the electron density perturbation $\widetilde{n}$ in Langmuir waves can be described by

$$\partial_t^2 \widetilde{n} + \omega_{p,0}^2 \widetilde{n} - 3v_T^2 \nabla^2 \widetilde{n} = -\nabla \cdot \boldsymbol{\eta}/m. \qquad (1)$$

Here, $\omega_{p,0}$ is the unperturbed plasma frequency, v_T is the thermal speed, $\boldsymbol{\eta} = -n\nabla\Phi$ is the ponderomotive force density (assume that $\boldsymbol{\eta}$ is of order $\widetilde{n}$), $\Phi = e^2|E_c|^2/(4m\omega^2)$ is the ponderomotive potential, e is the electron charge, and m is the electron mass.

Hint: *As an option, you may derive the linear equation for $\widetilde{n}$ as if were no EM wave, then explain why the EM wave enters simply as an effective force $\boldsymbol{\eta}$. To save time, consider the expression for Φ as given, but explain why that particular expression applies to this problem.*

(b) [**10 points**] Assume that ω satisfies the cold-plasma dispersion relation. Show that $\boldsymbol{\eta}$ can be expressed through the EM-wave action density $\mathcal{I}$ as follows:

$$\boldsymbol{\eta} = -\frac{\omega_p^2}{2}\nabla\left(\frac{\mathcal{I}}{\omega}\right). \qquad (2)$$

(c) [**7 points**] When v_T is negligible, Eq. (1) becomes an equation of a driven oscillator. Using this, *estimate* the EM-pulse length that maximizes the amplitude of the plasma wave that the pulse excites by the ponderomotive force (the so-called plasma wake).

(d) [**10 points**] If plasma interacts with many randomly-phased EM waves (and parametric effects are ignored), their ponderomotive forces add up. Then, Eq. (2) becomes

$$\boldsymbol{\eta}(t, \mathbf{x}) = -\frac{\omega_p^2}{2}\nabla \int \frac{F(t, \mathbf{x}, \mathbf{k})}{\omega(t, \mathbf{x}, \mathbf{k})}\, d^3k, \qquad (3)$$

where F is the photon distribution. Assuming that photons can be modeled as classical particles governed by ray equations, write the corresponding kinetic equation for F. Then adopt $F = F_0(\mathbf{k}) + \widetilde{F}(t, \mathbf{x}, \mathbf{k})$ and $\omega = \omega_0(\mathbf{k}) + \widetilde{\omega}(t, \mathbf{x}, \mathbf{k})$, where the tilded quantities are the linear perturbations of order $\widetilde{n}$. Assuming $\widetilde{n} \propto e^{-i\Omega t + i\mathbf{K}\cdot\mathbf{x}}$, show that

$$\widetilde{F} = -\frac{\widetilde{\omega}\mathbf{K}\cdot\nabla_{\mathbf{k}}F_0}{\Omega - \mathbf{K}\cdot\mathbf{v}_g}, \qquad \widetilde{\omega} = \frac{\omega_{p,0}^2}{2\omega_0}\frac{\widetilde{n}}{n_0}, \qquad (4)$$

where $\mathbf{v}_g$ is the photon unperturbed group velocity. (The wave vector of a Langmuir wave, $\mathbf{K}$, must not be confused with the wave vectors of EM waves $\mathbf{k}$.)

(e) [**8 points**] Using Eqs. (1), (3), and (4), derive the Langmuir wave dispersion for a given F_0. In general, is the frequency real or complex? Why?

II.2: Experimental Methods [15 points]

Consider a thin-walled glass sphere, 10 cm in radius, containing a uniform 1 eV Ar^+ plasma of density $10^{10} - 10^{12}$ cm^{-3}. One would like to measure its density with a microwave interferometer.

(a) [3 points] What frequency microwaves would be appropriate?

(b) [10 points] Sketch a simple homodyne Mach-Zehnder interferometer, naming at least 6 distinct components. (The same component can be used several times).

(c) [2 points] If the plasma were of chlorine instead argon, what complication might arise?

II.3: Applied Math [45 points]

Consider a model equation that describes a perturbed particle distribution function $y(x)e^{i\varphi}$ with $y(\pm\infty) = 0$, in an extended pitch-angle variable x:

$$\mathcal{C}_p[y] + ixy + 1 = 0, \tag{1}$$

where the bounce-averaged pitch-angle collisional operator is given by

$$\mathcal{C}_p[y] = \nu \frac{d}{dx}\left(\operatorname{sech}(x)\frac{dy}{dx}\right) \tag{2}$$

in the modeled geometry. You will be asked to study this model equation analytically and evaluate a quantity related to the torque $\mathcal{T}$ in the low collisionality limit:

$$\mathcal{T} = \lim_{\nu\to 0}\int_{-\infty}^{\infty} \mathcal{J} y(x)dx, \tag{3}$$

where $\mathcal{J}$ is just a constant related to the variation in the action. The following formulas (or definitions) may be helpful for analysis.

$$\delta(x) = \lim_{a\to 0}\frac{1}{\pi}\frac{a}{x^2+a^2}$$

$$\delta(x) = \lim_{a\to\infty}\frac{1}{2\pi}\int_{-a}^{a} e^{ixt}dt$$

$$\operatorname{sech}(x) \equiv \frac{2}{e^x+e^{-x}} = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots \quad \left(|x| < \frac{\pi}{2} \text{ for series}\right)$$

(a) [**5 points**] First try the Krook operator $\mathcal{C}_k[y] = -\nu y$ instead, and obtain $\mathcal{T}$.

(b) [**5 points**] Now with the pitch-angle operator $\mathcal{C}_p[y]$, take ν as a small parameter and use the boundary-layer theory to proceed with your analytic evaluation. Where is the boundary layer? Give the outer-layer equation and solution.

(c) [**5 points**] Look into only the homogeneous part of the equation. Use a Kruskal-Newton diagram to find the dominant balance for the inner layer, and the width of the layer.

(d) [**7 points**] It is however necessary to include the inhomogeneous part for the inner-layer equation to match the outer-layer solution. You may expect this from the Kruskal's philosophical principle of *maximal complexity* for asymptotic analysis. By rescaling $y \to Y$ as well as $x \to X$ with the small parameter ν, show that the inner-layer equation can be cast as the inhomogeneous Airy equation:

$$\frac{d^2Y}{dX^2} + iXY = -1. \tag{4}$$

- Problem continued on next page -

(e) **[8 points]** Using the Fourier-Laplace Kernel $K(X,t) = e^{iXt}$, show that a particular solution for the layer equation can be given by

$$Y_p(X) = \int_0^\infty e^{iXt - t^3/3} dt. \tag{5}$$

Hint: *Choose the end points of the integral path, in such a way to match the remainder of integration by parts to the inhomogeneous term.*

(f) **[10 points]** Find the path of steepest descent for the integral (5) on the complex t-plane, for $X \to +\infty$. Show that the dominant contribution comes from the integration near the end point $t = 0$, rather than a saddle point. Note that this is also true for $X \to -\infty$. Determine the large X asymptotic behavior of the particular layer solution $Y_p(X)$ and see if it matches the outer-layer solution in (b). Obtain a leading-order (in ν) uniform approximation to $y(x)$.

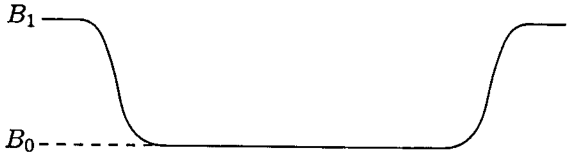
Hint: *Here you don't have to fix the movable saddle points with another variable transformation. Just assume that X is a fixed parameter. Also use the approximation $I = \int_C e^{\phi(X,t)} dt \sim \sqrt{-2\pi/\phi''(X,t_s)} e^{\phi(X,t_s)}$ to the integration through a saddle point $t = t_s$, if it is needed.*

(g) **[5 points]** Evaluate $\mathcal{T}$ using the uniform asymptotic approximation of $y(x)$ and compare your final answer with (a). This is a simplified example illustrating that the details of the collision operator are not important in the plateau regimes (*e.g. potato-plateau, superbanana-plateau*), as is well known in neoclassical transport theory, because the purpose of collisions is just to remove the singularity in the kinematics.

II.4: Irreversible Processes [50 points]

This problem has **two** parts, both of which must be completed for full credit. Part II is on the next page; mathematical formulae of possible utility are given at the end of the problem.

Part I. Consider a mirror machine consisting of a long, straight, axisymmetric solenoid with magnetic field B_0 and with magnetic mirrors at both ends with peak field strength $B_1 > B_0$, as shown in the figure below. An ion-electron plasma fills the mirror, with enough scale separation between the Larmor, bounce, and collision frequencies to guarantee a magnetized, gyrotropic plasma with distribution function $f = f(v, \xi)$, where v is the particle speed and $\xi \equiv v_{\parallel}/v$ is the cosine of the pitch angle.



(a) [5 points] Ignoring collisions and electric fields, derive a condition for particles to be confined within such a device. Write your answer in terms of ξ and the mirror ratio $R_{\rm m} \equiv B_1/B_0$.

(b) [12 points] Now introduce particle collisions. Suppose that the mirroring particles undergo pitch-angle scattering as described by the Lorentz collision operator

$$\mathcal{C}[f] = \frac{\nu(v)}{2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f}{\partial \xi} \equiv \nu(v)\mathcal{L}[f], \tag{1}$$

where

$$\nu(v) \equiv \frac{3\sqrt{\pi}}{4\tau_{\rm coll}}\left(\frac{v_{\rm th}}{v}\right)^3 \tag{2}$$

is the velocity-dependent collision frequency, $\tau_{\rm coll}$ is the appropriate collision timescale, and $v_{\rm th} \equiv (2T/m)^{1/2}$ is the thermal speed. Under the watchful eye of an experimentalist, the resulting pitch-angle diffusion and consequent loss of particles from the device is perfectly balanced by a source of plasma particles, such that equilibrium is maintained. Write down an equation for this equilibrium, specify suitable boundary conditions, and solve for the equilibrium distribution function $f_{\rm eq}(v, \xi)$. Take the particle source S to be mono-energetic and independent of ξ for ξ satisfying the condition derived in (a).

(c) [13 points] Calculate the average confinement time $\tau_{\rm c}$ for particles introduced into the mirror-trapped region. Then take the limits $R_{\rm m} \sim 1$ and $R_{\rm m} \gg 1$ of $\tau_{\rm c}$. In which limit are particles lost from the device faster? Explain your answer physically.

Hint: *If you are unable to answer part (b), explain how you would calculate $\tau_{\rm c}$ if given $f_{\rm eq}(v, \xi)$. Provide physical arguments for whether $R_{\rm m} \sim 1$ or $R_{\rm m} \gg 1$ leaks particles faster.*

(d) [2 points] Ions and electrons generally have different collision frequencies, and so they will try to collisionally leak from the device at different rates. An ambipolar electric field will thus be set up to electrostatically confine whichever species would otherwise leak faster. Briefly explain which species, ions or electrons, is confined primarily by this ambipolar electric field. (Take the ion and electrons temperatures to be comparable.)

Part II. Now consider the Braginskii problem of collisional heat transport in a magnetized, thermally stratified plasma, that is, one with $\rho \ll \lambda_{\rm mfp} \ll L$, where ρ denotes the particles' Larmor radii, $\lambda_{\rm mfp}$ is the collisional mean free path, and L is the gradient lengthscale. Take the zeroth-order distribution functions of the ions and electrons to be isotropic Maxwellians with non-uniform density $n(\boldsymbol{r})$, zero mean flow, and equal non-uniform temperature $T(\boldsymbol{r})$:

$$f_0(\boldsymbol{r}, v) = \frac{n}{\pi^{3/2} v_{\rm th}^3} \exp\left(-\frac{v^2}{v_{\rm th}^2}\right), \quad \text{where} \quad v_{\rm th}^2 \equiv \frac{2T}{m}. \tag{3}$$

The corresponding correction equation that determines the first-order (non-Maxwellian) distribution function f_1 in the Braginskii-Chapman-Enskog expansion is given by

$$\Omega \frac{\partial f_1}{\partial \vartheta} + C[f_1] = \boldsymbol{v} \cdot \left[\frac{\boldsymbol{R}}{nT} + \left(\frac{v^2}{v_{\rm th}^2} - \frac{5}{2}\right) \boldsymbol{\nabla} \ln T\right] f_0, \tag{4}$$

where Ω is the Larmor frequency, ϑ is the particle gyrophase, and $\boldsymbol{R}$ is the friction force.

(e) [13 points] Solve equation (4) to obtain the parallel heat flux $\boldsymbol{q}_\parallel$ directed along the magnetic-field direction $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}/B$. For simplicity, take the collision operator C to be the Lorentz collision operator, $\nu(v)\mathcal{L}$, defined by equation (1). You may want to recall that the collision frequency $\nu(v) \propto v^{-3}$ (see equation 2) and that the Legendre polynomials $P_\ell(\xi)$ are eigenfunctions of the operator $\mathcal{L}$ with eigenvalues $-\ell(\ell+1)/2$.

Hint: *You do not need the full solution to equation (4) to calculate $\boldsymbol{q}_\parallel$, nor do you need the precise definition of $\boldsymbol{R}$ (unless you've forgotten some constraints on $f_1 \ldots$).*

(f) [5 points] In order for equation (3) to be self-consistent, something must balance the implied pressure gradient to give hydrostatic equilibrium. For example, this might be accomplished by a fixed gravitational field. But as conduction relaxes the temperature gradient, and hence the pressure gradient, the system will produce flows in response. That is, unless the temperature has a particular profile. What is it? For simplicity, take the magnetic field to be uniform and aligned with the temperature gradient.

<u>Possibly useless information:</u>

$$\ln(1+x) \simeq x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \quad \text{for } |x| \ll 1$$

$$\int \mathrm{d}x \, \ln(1-x^2) = x\ln(1-x^2) - 2x + \ln\left(\frac{1+x}{1-x}\right)$$

$$\int_{-1}^{1} \mathrm{d}x \, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\int_0^\infty \mathrm{d}x \, x^k \, \mathrm{e}^{-x} \equiv \Gamma(k+1) \;\; (= k! \text{ for integer } k \geq 0)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}, \quad \Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$$

II.5: MHD [20 points]

In this problem, you will be asked about properties of an ideal, rotating MHD plasma whose dynamics are described by

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} = -\frac{\boldsymbol{\nabla} P}{\rho} - 2\boldsymbol{\Omega} \times \mathbf{V} + \frac{(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}}{\mu_0 \rho}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{B}), \tag{2}$$

where $\boldsymbol{\Omega}$ is angular velocity and other symbols have their conventional meanings. You can assume incompressibility and uniform density for simplicity and use the vector identity $\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla} \cdot \mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla} \cdot \mathbf{a}) + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b}$, to answer the following questions.

(a) [10 points] In the limit of a rapid, uniform rotation $\boldsymbol{\Omega}_0$ without magnetic field, the geostrophic balance can be maintained. Prove that such a steady flow, $\mathbf{V}_0$, must be uniform in the direction of $\boldsymbol{\Omega}_0$ (*Taylor-Proudman Theorem*). What kind of waves can be supported in such flows when perturbed?

(b) [10 points] When a uniform magnetic field, $\mathbf{B}_0$, is imposed to the above flow along $\boldsymbol{\Omega}_0$, show that a perturbed magnetic field $\widetilde{\mathbf{B}}$, can grow out of the variation along $\mathbf{B}_0$ of a perturbed flow, $\widetilde{\mathbf{V}}$ (the ω effect). In which direction does this growing magnetic field point? How does magnetic field modify the waves?