DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

May 21, 2018

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name and the question number on the title page of each booklet.
- When you do not have time to put answers into forms that satisfy you, indicate ***specifically*** how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem ___" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

Contents

Page

I.1: General Plasma Physics [35 points]

Consider electrons in a magnetic mirror machine with mirror ratio R and mirror axis in the $\hat{z}$ direction. The electrons are sufficiently dilute that collisions are completely ignorable. Similarly, any self-generated electric or magnetic fields within the plasma are completely ignorable.

(a) [**6 points**] Derive the trapping condition in midplane ($z = 0$) energy coordinates, $W_{\perp 0}$ and $W_{\| 0}$, respectively the perpendicular and parallel energies as the midplane is crossed. Sketch in $W_{\perp 0}$-$W_{\| 0}$ space the region of trapped electrons.

(b) [**5 points**] Suppose that the magnetic field near the axis can be approximated as

$$\mathbf{B} = \begin{cases} B_0(1 + z^2/L^2)\hat{z}, & \text{if } z^2 < c^2L^2 \\ B_0(1 + c^2)\hat{z}, & \text{if } z^2 > c^2L^2. \end{cases}$$

Show that the turning points for trapped electrons obey: $z_T^2/L^2 = W_{\| 0}/W_{\perp 0}$. Note that one can write $R = 1 + c^2$.

(c) [**8 points**] Now suppose that $B_0 = B_0(t)$ is a very slowly varying function of time, such that it is monotonically increasing or decreasing in going from $B_0(t = 0) = B_{0i}$ to $B_0(t = t_f) = B_{0f}$, where $B_{0f}/B_{0i} = \beta$. Calculate the changes in the perpendicular and parallel midplane energies. Note: by slowly varying we mean slow enough that particle motion can be considered to be adiabatic.

(d) [**8 points**] For what midplane coordinates (as a function of β) will electrons initially trapped become untrapped as a result of the slowly changing $B_0 = B_0(t)$. Sketch in $W_{\perp 0}$-$W_{\| 0}$ space the region of initially trapped electrons that was initially trapped that then became untrapped.

(e) [**8 points**] Suppose that β is given, but it is possible to choose $B_0(t)$ to slowly oscillate between minimum $B_{\min}$ and maximum $B_{\max} = \beta B_{\min}$. Suppose further that the time it takes to go from $B_{\min}$ to $B_{\max}$ is τ_{up}, while the time it takes to go from $B_{\max}$ to $B_{\min}$ is τ_{down}. Finally, suppose that $B(t) = B_{\min}$ can be held constant for duration $\tau_{\min}$, while $B(t) = B_{\max}$ can be held constant for duration $\tau_{\max}$. How might you arrange the times τ_{up}, τ_{down}, $\tau_{\min}$ and $\tau_{\max}$ to **maximize** the loss (over many cycles) of electrons from the mirror machine? Note: You may assume now that these times are not necessarily very long (or that the corresponding rates need not be very slow). However, these times must all be very long compared to an electron cyclotron period $1/\Omega$, where $\Omega = eB/m$.

I.2: MHD [15 points]

Consider linear Alfvén waves propagating along a uniform magnetic field $\mathbf{B} = B_0\hat{z}$ in cold, incompressible, homogeneous plasma of constant density ρ.

(a) [**6 points**] Show that the wave equation obeyed by the linear fluctuation $\mathbf{B}_1$ is given by:

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho}\frac{\partial^2 \mathbf{B}_1}{\partial z^2}.$$

(b) [**5 points**] Calculate the Poynting flux in terms of the fluctuations $\mathbf{v}_1$ and $\mathbf{B}_1$.

(c) [**4 points**] What is the relative magnitude of the magnetic and kinetic energies of the fluctuations?

I.3: Waves and Instabilities [50 points]

Consider a stationary R wave propagating parallel to static magnetic field $\mathbf{B}_0$ in a cold collisionless plasma. Assume that the wave frequency ω is large enough such that the ion susceptibility is negligible. Also assume that the electron density can be considered constant and the electron cyclotron frequency can be approximated as $|\Omega_e| \approx \omega(1 - z/L)$, where L is a constant and z is a coordinate along $\mathbf{B}_0$. (The dependence on the transverse coordinates is assumed negligible.) Below, you are asked to study transformations of this wave that result from the inhomogeneity of $\mathbf{B}_0$ near the electron cyclotron resonance (ECR).

Note: Solving this problem does *not* require cumbersome calculations or applications of advanced machinery like the Weyl transform, WKB approximation, and special functions.

(a) [12 points] Starting with Maxwell's equations, show that the wave complex amplitude $\mathcal{E}$ (which is some scalar linear combination of $E_{x,y,z}$) can be described by

$$\frac{\partial^2 \mathcal{E}}{\partial z^2} + \frac{\omega^2}{c^2}\left(1 - \frac{\alpha L}{z}\right)\mathcal{E} = 0, \quad \alpha = \frac{\omega_{pe}^2}{\omega^2}. \tag{1}$$

In doing so, you might find the following formula relevant:

$$\hat{\boldsymbol{\epsilon}} = \begin{pmatrix} 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} & -i\sum_s \frac{\Omega_s}{\omega}\frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} & 0 \\ i\sum_s \frac{\Omega_s}{\omega}\frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} & 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} & 0 \\ 0 & 0 & 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \end{pmatrix}.$$

You do *not* have to derive it, but please explain why the expression for the homogeneous-plasma dielectric tensor is applicable in this problem even though $\mathbf{B}_0$ is inhomogeneous. Not all plasmas have this property; which plasmas do?

(b) [13 points] Assuming the geometrical-optics approximation and using Eq. (1), derive $k(z)$, where k is (the z-component of) the wave vector. Identify the cutoff(s) and resonance(s). Plot both branches of k as functions of $\xi \equiv z/L$ at small α. Argue that there are two separate regions where mode conversion occurs. Describe what happens to a wave incident from the high-B_0 side (i.e., from $z < 0$) and from the low-B_0 side (i.e., from $z > 0$). Indicate the directions of the action fluxes on your plots.

- Problem continued on next page -

(c) [**10 points**] Suppose a stationary wave incident from the high-B_0 side. Assuming a given amplitude $\mathcal{E}(z_0)$ at some $z_0 < 0$ and given transmission coefficient $|\mathcal{T}|^2 < 1$ for the wave-action flux [which you will be asked to estimate later in part (d)], calculate and sketch $\mathcal{E}(z)$ within the geometrical-optics approximation at all $z < 0$ and $z > 0$. Where does the geometrical-optics approximation fail? What happens to the part of the radiation that is *not* transmitted?

(d) [**15 points**] Finally, try to *estimate* the coefficient $|\mathcal{T}|^2$ that was considered in part (c) as given. In order to do so, Taylor-expand $k(z)$ to the first order in α and show that $\xi\eta = \text{const}$, where $\eta \equiv (k - \omega/c)L$ is a rescaled wave number. Introduce new phase-space coordinates (q, p) that correspond to the clockwise rotation of the (ξ, η) plane by 45°. Plot the resulting canonical form of the dispersion curves in the mode-conversion region. As you might recall, $\ln \mathcal{T} \approx -\pi/\mu$, where μ is a dimensionless parameter that has a simple physical meaning for a dispersion relation in the canonical form. Explain this physical meaning and estimate μ. Using your result, formulate the condition under which no substantial energy is transferred across the cutoff region. Also comment on the mode conversion efficiency in the low-density limit.

I.4: Kinetic Theory [15 points]

Consider the collisionless Vlasov equation for electrons and ions interacting with an electrostatic potential from the Poisson equation (and no magnetic field). Show that total momentum is conserved. (You can use simple boundary conditions to simplify the boundary terms.)

Hint: Near the end of the calculation, you may find it useful to express things in terms of the potential in order to do certain manipulations.

I.5: Experimental Methods [30 points]

Consider an interferometer at wavelength, $\lambda = 10.6\mu\text{m}$ (CO2 laser), and operates in the presence of spurious vibrations of the optical components. To compensate for these vibrations, interferometry is performed simultaneously using the same optical components at a wavelength of $\lambda = 0.633\mu\text{m}$ (HeNe laser). The HeNe interferometer is affected much less than the CO2 interferometer by the plasma phase shift, but still somewhat. If $\omega \gg \omega_p$ for both wavelengths:

(a) [**10 points**] Derive an expression for the plasma line integrated density, $\int N_e dl$, in terms of the phase shifts, Φ_C and Φ_{He}, of the two interferometers.

(b) [**10 points**] If Φ_{He} can be measured with an accuracy of $\pm\pi$, what uncertainty does this introduce into the plasma density measurement?

(c) [**10 points**] Thus evaluate the fractional error in measuring a 1m thick plasma of density 10^{14}cm^{-3}, assuming Φ_C is measured exactly.

I.6: Applied Math [20 points]

Newcomb's analysis (1960) shows that an exponentially growing perturbation exists in an ideal cylindrical plasma if a radial displacement ξ, minimizing the variation of the potential energy δW, changes sign between successive singular radial points. Consider the Euler-Lagrange equation for minimizing δW,

$$\frac{d}{dx}\left[\left(q'x + \frac{1}{2}q''x^2\right)^2 \frac{d\xi}{dx}\right] - \left(p' + q'x\right)\xi = 0,$$

near a singular surface at $r = r_s$, where $x \equiv r - r_s$. Here the coefficients are all simplified except $q' = (dq/dr)_{r=r_s}$, $q'' = (d^2q/dr^2)_{r=r_s}$, and $p' = (dp/dr)_{r=r_s}$, where q is the safety factor and p is the pressure. The goal is to find the solution behavior asymptotic to $x \to +0$ and see if $\xi(x)$ is oscillatory, as it means an instability due to Newcomb's conclusion.

(a) [**8 points**] Assume $p' \neq 0$ and $q' \neq 0$. Classify the $x = 0$ point and find the leading behaviors of the two linearly independent solutions for $x \to +0$. Note that the $q''x^2$ term can be ignored here. Give the necessary condition for stability. This is essentially the Suydam's criterion (1958) against localized interchange modes.

(b) [**12 points**] Consider a special case of $q' = 0$, where the $q''x^2$ term is no longer ignorable, while still $p' \neq 0$. Classify the $x = 0$ point and find the leading behaviors of the two linearly independent solutions for $x \to +0$. Show the instability expected, unless the pressure increases along the radius, i. e. $p' > 0$.

I.7: Experimental Methods [15 points]

For a gas-discharge producing a partially ionized plasma, using equations for spatially averaged particle balance and power balance, determine a plasma response to changes of the discharge input parameters, specifically:

(a) [**6 points**] If one increases the input power to this discharge, while keeping the gas pressure constant, what should happen to the plasma density?

(b) [**9 points**] If one increases the gas pressure, while keeping the power constant, what should happen to the electron temperature?

To answer these questions,

- Consider partially-ionized plasma $n_e/(n_e + n_n) \ll 1$, where n_e and n_n are the densities of electrons and neutrals, respectively.
- Assume Maxwellian electron energy distribution function, hot electrons and cold ions ($T_e > T_i$).

Also, the following are global model balances to use:

- Particle balance: volume ionization = ambipolar plasma flux to the wall per second. Approximate the ionization frequency as $\nu_{iz} \sim n_n \exp(-E_{iz}/T_e)$, where n_n is neutral density, E_{iz}, T_e are the ionization potential and the electron temperature in eV.
- Power balance: input electric power = power losses in plasma on ionization and wall losses. Assume that the total energy losses are expressed as the sum of energy loss per electron-ion pair created, mean energy lost per electron lost to the wall, and mean energy lost per ion lost to the wall. That is,

$$\varepsilon_T = \varepsilon_{iz}(T_e) + 2T_e + \varepsilon_{ion}(T_e).$$

DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

May 22, 2018

9:00 a.m. – 1:00 p.m.

- Answer all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
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Contents

Page

II.1: MHD [35 points]

In this problem, you will be asked to derive force-free fields as states of minimum magnetic energy, $W = \int (B^2/2\mu_0) dV$, and to evaluate their stability by using the energy principle.

(a) [**5 points**] When the plasma contained in V is displaced by a small $\boldsymbol{\xi}$, the perturbed magnetic field is given by $\boldsymbol{B}_1 = \nabla \times \boldsymbol{A}_1$, where $\boldsymbol{A}_1 \equiv (\boldsymbol{\xi} \times \boldsymbol{B})$ is a perturbed vector potential. What is the perturbed magnetic energy? Assuming that $\boldsymbol{\xi}$ vanishes on the boundary of V, show that the condition for the perturbed magnetic energy to vanish for any $\boldsymbol{\xi}$ leads to a force-free field.

Hint: you can use $\boldsymbol{\nabla} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{a}) - \boldsymbol{a} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b})$ *and* $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{c} \times \boldsymbol{a}$.

(b) [**10 points**] Derive a second force-free field by using the variational principle with a Lagrange multiplier (λ) to minimize W while conserving the total magnetic helicity $K \equiv \int \boldsymbol{A} \cdot \boldsymbol{B} dV$, where $\boldsymbol{A}$ is a vector potential of magnetic field $\boldsymbol{B}$.

Hint: Express the variation of W *in terms of* $\delta \boldsymbol{A}$ *which can be assumed to vanish on the boundary of* V.

(c) [**5 points**] What is the difference between these two force-free fields?

The energy integral due to displacement $\boldsymbol{\xi}$ is given by

$$\begin{aligned}\delta W \; = \; & \frac{1}{2} \int dV \left[\frac{|\boldsymbol{Q}_\perp|^2}{\mu_0} + \frac{B^2}{\mu_0} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + 2 \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} \right)^2 + \gamma p \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right)^2 \right. \\ & \left. -2 \left(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p \right) \left(\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa} \right) - \frac{\boldsymbol{j} \cdot \boldsymbol{B}}{B^2} \left(\boldsymbol{\xi}_\perp^* \times \boldsymbol{B} \right) \cdot \boldsymbol{Q}_\perp \right],\end{aligned}$$

where $\boldsymbol{Q} \equiv \boldsymbol{B}_1$, $\boldsymbol{\kappa} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b}$ and $\boldsymbol{b} = \boldsymbol{B}/B$.

(d) [**5 points**] What are the possible signs of each term for each of the two force-free fields obtained above?

(e) [**10 points**] Evaluate stability of these two force-free fields.

Hint: For the second force-free field, you can use conservation of magnetic helicity of the perturbed field, $\int \boldsymbol{A}_1 \cdot \boldsymbol{B}_1 dV = 0$.

II.2: Waves and Instabilities [10 points]

Over the last decade, several radio telescopes detected multiple "fast radio bursts" (FRBs), which are millisecond-range broadband flashes with frequency chirping described by

$$\frac{d}{dt}\left(\frac{1}{\omega^2}\right) \approx \text{const.}$$

Many of these FRBs, particularly those detected by the Parkes telescope in Australia, were recently identified as artifacts caused by a microwave oven in the staff kitchen.[1] But the other FRBs are still believed to be generated by instantaneous broadband sources of (unknown) extragalactic origin, and their chirping is believed to be caused by the wave dispersion in space plasma. Show that the above equation for ω is consistent with this hypothesis.

[1] E. Petroff *et al.*, *Identifying the source of perytons at the Parkes radio telescope*, MNRAS 451, 3933 (2015).

II.3: Irreversible Processes [50 points]

This problem has **two** parts, **both** of which must be completed for full credit. Part B is on the next page. Mathematical formulae of possible utility are given at the end of the problem. *Note that each question,* (a)–(f)*, may be answered independently of one another!*

Part A. Consider an initial beam distribution of charged particles with a single velocity $\boldsymbol{v}_0$, i.e. $f(t=0,\boldsymbol{v}) = n\,\delta(\boldsymbol{v}-\boldsymbol{v}_0)$. In a spherical velocity-space coordinate system,

$$f(t=0,\boldsymbol{v}) = \frac{n}{2\pi v_0^2}\,\delta(v-v_0)\,\delta(\xi-1), \tag{1}$$

where $\xi \equiv \cos\theta$ is the cosine of the pitch angle. These particles are subject to pitch-angle scattering by the Lorentz collision operator:

$$C[f] = \frac{\nu}{2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) = \frac{\nu}{2}\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial f}{\partial\xi}\right] \equiv \nu\mathcal{L}[f], \tag{2}$$

where ν is the collision frequency (assumed to have the customary dependences on particle mass and on plasma density and temperature, but no dependence on particle velocity).

(a) [**7 points**] For short times ($\nu t \ll 1$) such that the particles have diffused only a little bit in θ, show that

$$f(\nu t \ll 1, v, \theta) \approx \frac{n}{2\pi v_0^2}\,\delta(v-v_0)\,\frac{\exp(-\theta^2/2\nu t)}{\nu t}, \tag{3}$$

Use this to further show that the mean angular spread for short times obeys the random-walk scaling $\langle\theta^2\rangle \approx 2\nu t$.

Hint: The Green's function for the diffusion equation $\partial_t u - \nabla^2 u = S$ in an infinite, 2D, cylindrical domain is $G(t,R;t',R') = (4\pi\tau)^{-1}\exp[-(R-R')^2/4\tau]$ for $\tau \equiv t-t' > 0$.

(b) [**10 points**] Now consider later times ($\nu t \gtrsim 1$), for which $\langle\theta^2\rangle \sim 1$. Expand $f(t,v,\xi)$ in Legendre polynomials $P_\ell(\xi)$ to obtain the solution

$$f(t,v,\xi) = \frac{n}{2\pi v_0^2}\,\delta(v-v_0)\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(\xi)\exp\left[-\frac{\nu t}{2}\,\ell(\ell+1)\right]. \tag{4}$$

(Recall $\mathcal{L}P_\ell(\xi) = -[\ell(\ell+1)/2]P_\ell(\xi)$.) What does equation (4) become when $\nu t \to \infty$? Does your answer make sense physically? (Simply writing "yes" or "no" is not enough.)

(c) [**13 points**] Use equation (4) to calculate the expectation values $\langle v_\parallel(t)\rangle$, $\langle v_\parallel^2(t)\rangle$, and $\langle v_\perp^2(t)\rangle$. What are the corresponding Fokker-Planck coefficients, $\boldsymbol{A}(\boldsymbol{v})$ and $\mathsf{B}(\boldsymbol{v})$, representing drag and diffusion? Provide a physical explanation of the result.
If you cannot explicitly calculate these expectation values, you may still receive partial credit for (i) *stating the definitions of the Fokker–Planck coefficients and/or* (ii) *anticipating the correct forms of $\boldsymbol{A}(\boldsymbol{v})$ and $\mathsf{B}(\boldsymbol{v})$ and justifying them in physical terms.*

Part B. Now consider the Braginskii problem of collisional momentum transport in a magnetized ion–electron plasma, that is, one with $\rho \ll \lambda_{\rm mfp} \ll L$, where ρ denotes the particles' Larmor radii, $\lambda_{\rm mfp}$ is the collisional mean free path, and L is some gradient lengthscale. Ignore the electrons (you'll state why in question (e)). Take the zeroth-order distribution function of the ions to be an isotropic Maxwellian with uniform density n, non-uniform incompressible mean velocity $\boldsymbol{u}(t,\boldsymbol{r})$, and non-uniform temperature $T(t,\boldsymbol{r})$:

$$f_0(t,\boldsymbol{r},v) = \frac{n}{\pi^{3/2}\,v_{\rm th}^3}\,\exp\left(-\frac{w^2}{v_{\rm th}^2}\right), \quad \text{where} \quad v_{\rm th}^2 \equiv \frac{2T(t,\boldsymbol{r})}{m} \tag{5}$$

and $\boldsymbol{w} \equiv \boldsymbol{v} - \boldsymbol{u}(t,\boldsymbol{r})$ is the velocity of the ions peculiar to their mean flow. The corresponding correction equation that determines the first-order (non-Maxwellian) distribution function f_1 in the Braginskii–Chapman–Enskog expansion is given by

$$\Omega\,\frac{\partial f_1}{\partial \vartheta} + \mathcal{C}[f_1] = f_0\left[\left(\frac{w^2}{v_{\rm th}^2} - \frac{5}{2}\right)\boldsymbol{w}\boldsymbol{\cdot}\nabla\ln T + \frac{2\boldsymbol{w}\boldsymbol{w}}{v_{\rm th}^2}\boldsymbol{:}\nabla\boldsymbol{u}\right], \tag{6}$$

where Ω is the Larmor frequency and ϑ is the particle gyrophase.

(d) [13 points] Solve equation (6) to obtain the parallel viscous stress tensor $\boldsymbol{\Pi}_\parallel$. For simplicity, take the collision operator $\mathcal{C}$ to be the v-independent Lorentz collision operator defined by equation (2). Once again, recall that the Legendre polynomials $P_\ell(\xi)$ are eigenfunctions of the operator $\mathcal{L}$ with eigenvalues $-\ell(\ell+1)/2$.

Hint: You do not need the full solution of equation (6) to calculate $\boldsymbol{\Pi}_\parallel$*!*

(e) [2 points] If you were to compute the parallel viscous stress for the electrons, you would find it to be smaller than that of the ions (provided the ions' and electrons' mean velocities and temperatures are similar). By what factor is it smaller?

(f) [5 points] Write down an evolution equation for the temperature of a fluid element, including parallel viscous heating. (Recall that the plasma is incompressible.) How can a magnetized, collisional plasma avoid parallel viscous heating? If such an arrangement cannot be achieved, how does the rate-of-change of temperature in a fluid element (i.e., $\mathrm{d}T/\mathrm{d}t \equiv \partial T/\partial t + \boldsymbol{u}\boldsymbol{\cdot}\nabla T$) depend on that fluid element's temperature?

<u>Possibly useless information:</u>

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} \text{ in cylindrical-polar coordinates } (R,\phi)$$

$$\int_{-1}^{1}\mathrm{d}x\,P_\ell(x)P_{\ell'}(x) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}, \quad \delta(\xi-\xi_0) = \sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(\xi)P_\ell(\xi_0)$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$\int_0^\infty \mathrm{d}x\,x^k\,\mathrm{e}^{-x} \equiv \Gamma(k+1)\ \big(= k!\text{ for integer } k \geq 0\big)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}, \quad \Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$$

II.4: Elementary Plasma Physics [15 points]

(a) [**9 points**] Derive the particle diffusion coefficient of a low-β, infinite (in y and z) plasma slab (warm electrons & cool ions) expanding slowly in the x direction through a pressure $p(x)$ and a z-directed magnetic field $\mathbf{B} = B(x)\hat{z}$.

You can do this with a single-particle (interacting with many particles) model or fluid equations. The diffusion is ambipolar. If you choose the fluid approach, then you can make the assumptions:

- The diffusion is slow, hence inertial effects are unimportant.
- Simplified Ohm's Law: $\mathbf{v} \times \mathbf{B}/c = \eta \mathbf{J}$.
- The B field is constant is time
- There is no current parallel to **B**
- There is no current in the x-direction
- The plasma is isothermal

(b) [**6 points**] What happens to the energy released in the expansion?

II.5: Neoclassical & Anomalous Transport [35 points]

Part A. This problem deals with "neoclassical" properties of a plasma for a simplified model axisymmetric toroidal plasma.

(a) [**5 points**] Give the dimensionless measure of collisionality characterizing the long-mean-free-path "banana" regime and briefly explain the terms used.

(b) [**5 points**] State (without proof) how the classical (Spitzer) conductivity is modified in the "banana" regime.

(c) [**5 points**] Explain the physics of the "bootstrap current" and give two reasons why it is important.

Part B. Electrostatic drift waves (and associated instabilities) are often invoked when neoclassical theory proves inadequate to account for the higher levels of transport observed in toroidal experiments.

(d) [**15 points**] Estimate the perturbed ion density response for "cold fluid" ions in a simple slab geometry with a density gradient (including the 2-nd order terms in wave number, of order $(k_\perp \rho_s)^2$). Using quasineutrality, combine this with the perturbed electron density response for kinetic electrons in the lowest-order "adiabatic" limit, to obtain the electrostatic drift-wave dispersion relation. Justify the use of the "quasineutrality condition".

(e) [**5 points**] Compare the drift-wave frequency with the cyclotron frequency to explain why drift waves are not contained in the standard ideal MHD equations.

II.6: General Plasma Physics [15 points]

In a uniform cold neutral plasma, assume that negatively charge species has a finite size with a fixed shape function $s(d)$, where d is the distance from the center of the particle. The shape function is an even function, i.e., $s(d) = s(-d)$, and has a finite support, i.e., $s(d) = 0$ for d large enough. It describes the distribution of charge of the finite size particle such that the charge density at one spatial location due to the negative species is

$$\rho(x,t) = q \int n(x',t)s(x-x')dx',$$

where $n(x,t)$ is the number density of the centers of the particles. Assume that the positively charged species is heavy and provides a fixed positive background. Consider the one dimensional electrostatic plasma oscillation.

Show that the dispersion relation for the dynamics is

$$\omega^2 = \omega_p^2 S^2(k),$$
$$\omega_p^2 \equiv \frac{4\pi n q^2}{m},$$

where n is density of the centers of the negative species and $S(k)$ is the Fourier spectrum of the shape function $s(x)$.

Hint: You may need to know the following fact about the Fourier transform of the convolution of two functions. If $h(z) = \int f(x)g(z-x)dx$, then $H(k) = F(k)G(k)$. Here, $H(k)$, $F(k)$, and $G(k)$ are the Fourier spectrum of $h(x)$, $f(x)$, and $g(k)$, respectively.

II.7: Experimental Methods [20 points]

Inertial fusion plasma is confined by transiently compressing the fuel pellet. The radius, R, of the fuel pellet is compressed inward to the center at the sound speed, C_s. The radius during the inward compression is expressed as $R(t) = R_0 - C_s t$, where R_0 is the fuel radius at time $t = 0$. The time needed for compression is $\sim T_0 = R_0/C_s$.

(a) [12 points] Determine the expression for required confinement time, τ, needed for inertial fusion from the integral, $\tau = \int V(t)/V_0 dt$, where the integral is from $t = 0$ to $t = T_0 = R_0/C_s$. $V(t)$ is the volume of the fuel pellet as a function of time, and V_0 is the initial volume of the fuel pellet $t = 0$.

(b) [8 points] If the sound speed for compression condition is $C_s = 10^8$cm/s and the fuel radius is 0.5mm, what is the confinement time, τ?