DEPARTMENT OF ASTROPHYSICAL SCIENCES, PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART I

May 8, 2023

9:00 a.m. – 1:00 p.m.

- Today's exam (Part I) contains 6 problems on pages 2–8. Attempt all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
- Start each numbered problem in a new test booklet. Put your name and the question number on the title page of each booklet.
- When you do not have time to put answers into forms that satisfy you, indicate as specifically as you can how you would proceed if more time were available. If you do not attempt a particular problem, write on the booklet "I have not attempted Problem ___" and sign your name.
- All work on this examination must be independent. No assistance from other persons is permitted.
- An NRL formulary is permitted. No other aids (books, calculators, etc.) are allowed.

Contents

Page

I.1 General Plasma Physics [45 points]

(a) [6 pts] Suppose a magnetic mirror with mirror axis in the $\hat{z}$ direction, with midplane $z = 0$ and the ends of the mirror are at $z = z_1$ and $z = -z_1$. The midplane magnetic field $B(z = 0) = B_0$ is the minimum magnetic field and (for this part) the largest magnetic field is at the endpoints, i.e. $B(z = \pm z_1) = B_1 = B_{max}$. Derive the trapping condition in terms of the mirror ratio $R \equiv B_1/B_0$ and the perpendicular and parallel energies of electrons as the midplane $z = 0$ is crossed, namely $W_{\perp 0}$ and $W_{\parallel 0}$.

(b) [3 pts] Sketch the trapping condition in the $W_{\perp 0}$-$W_{\parallel 0}$ plane.

Now suppose a very different kind of magnetic trap for electrons, but now the largest magnetic field is the midplane magnetic field $B(z = 0) = B_0$. while the smallest magnetic field is at the endpoints $z = a$ and $z = -a$. (The z_1 parameter doesn't appear for this part of the problem.) To provide electron confinement, the magnetic trap is held at positive potential. The magnetic field near the axis can be put as

$$\mathbf{B} = \begin{cases} B_0(1 - z^2/L^2)\hat{z}, & \text{if } z^2 < a^2 \\ B_0(1 - a^2/L^2)\hat{z}, & \text{if } z^2 > a^2, \end{cases}$$

where $a^2/L^2 < 1$, and that the electric potential near the axis can be approximated as

$$\phi = \begin{cases} \phi_0(1 - z^2/a^2), & \text{if } z^2 < a^2 \\ 0, & \text{if } z^2 > a^2. \end{cases}$$

Electrons are considered trapped if they are confined to $z^2 < a^2$.

(c) [8 pts] Show that the parallel energy of trapped electrons can be put in the form

$$W_{\parallel}(z) = W_{\parallel 0}\left(1 - z^2/z_T^2\right).$$

Express the turning points z_T in terms of $W_{\perp 0}$, $W_{\parallel 0}$, ϕ_0, a and L.

(d) [4 pts] Sketch the trapping condition in the $W_{\perp 0}$-$W_{\parallel 0}$ plane.

(e) [5 pts] Suppose that if electrons pitch angle scatter (no energy change), they can be lost through the trapped-passing boundary. In your sketch of the trapping condition, show the triangular region of electrons that can be lost through repeated pitch angle scattering. Show that the area A of this region is:

$$A = \frac{1}{2}\left(e\phi_0\right)^2\left(L^2/a^2 - 1\right).$$

Suppose now that the magnetic field is not constant, but instead changes very slowly in time, eventually changing by a factor α. In other words, if $B_0 \to \alpha B_0$

(f) [3 pts] How does the perpendicular midplane energy $W_{\perp 0}$ change?

(g) [9 pts] How does the parallel midplane energy $W_{\parallel 0}$ change?

(h) [7 pts] In terms of the initial ($t = 0$) $W_{\perp 0}$ and $W_{\parallel 0}$, write down the condition for electrons that are initially trapped but then detrapped. Are there any such electrons for $\alpha > 1$? Are there any such electrons for $\alpha < 1$?

Hint: You may wish to use the integral

$$\int_a^b \left[(s-a)(b-s)\right]^{1/2} ds = \frac{\pi}{8}(b-a)^2$$

I.2 Waves and Instabilities: Ion Acoustic Instability [40 points]

In this problem, you are asked to explore electrostatic oscillations in three-dimensional homogeneous collisionless electron-ion plasma. The usual notation will be assumed.

(a) [**10 points**] Show that such oscillations satisfy the following dispersion relation:

$$0 = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int \frac{\boldsymbol{k} \cdot \partial_{\boldsymbol{v}} f_{s0}(\boldsymbol{v})}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega} \, \mathrm{d}\boldsymbol{v} \equiv \epsilon(\omega, \boldsymbol{k}). \tag{1}$$

Ignore the subtleties associated with the resonant denominator, but do explain how the integral in (1) is interpreted in the rigorous theory.

Hint: There is no need to introduce the plasma conductivity and general dispersion theory here. Use the minimum set of equations that you actually need in this specific problem.

(b) [**10 points**] Assume that ions are cold and electrons are Maxwellian. Allow for a small average electron velocity $\boldsymbol{V}$ (needed in part (c)), but ignore the contribution of resonant particles in this part of the problem. By taking the appropriate limit of (1), show that

$$\epsilon(\omega, \boldsymbol{k}) \approx \epsilon_{\mathrm{r}}(\omega, \boldsymbol{k}) \equiv 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{1}{k^2 \lambda_{\mathrm{De}}^2}, \tag{2}$$

where λ_{De} is the electron Debye length. Derive the corresponding dispersion relation of ion acoustic waves.

(c) [**17 points**] Assume that $\boldsymbol{V}$ is of the order of the ion sound speed. Using your results from part (b), show that the growth rate of ion acoustic waves is given by

$$\gamma = \sqrt{\frac{\pi}{8} \frac{Z_i m_e}{m_i}} \frac{\boldsymbol{k} \cdot \boldsymbol{V} - \omega}{(1 + k^2 \lambda_{\mathrm{De}}^2)^{3/2}}. \tag{3}$$

(d) [**3 points**] Find the angles between $\boldsymbol{k}$ and $\boldsymbol{V}$ at which ion acoustic waves are unstable at $k\lambda_{\mathrm{De}} \ll 1$. Qualitatively, what stabilizes these waves at large k?

I.3 Experimental Methods: Langmuir probe (30)

1. An idealized single Langmuir-probe characteristic is shown below (figure 1) for a cool plasma composed of singly charged ions and electrons. The probe is a 1 cm^2 paddle. No magnetic field is present.
   a) Indicate (on the characteristic) the floating and space potentials. What are their values? (3)
   b) Indicate the ion and electron saturation currents. (3)
   c) What is the (dominant) species of ions found in this plasma? Explain how you arrived at that conclusion. (4)
   d) Make an estimate of the electron temperature and explain your reasoning. (4)
   e) Make an estimate of the electron density and explain your reasoning. (4)

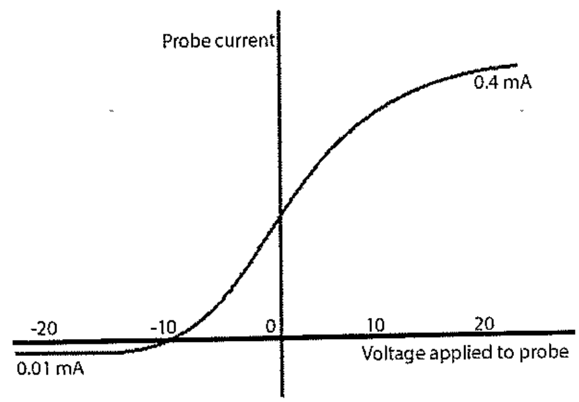

Figure 1

2. The characteristic of a spherical Langmuir probe of radius 0.1 cm is shown below (Figure 2) for a plasma with density and temperature similar to that in Part 1 (above): (12)
   a) Give at least two explanations for the differences in shape from the idealized probe characteristic (above) for region A and one reason for region B. Explain your reasons.

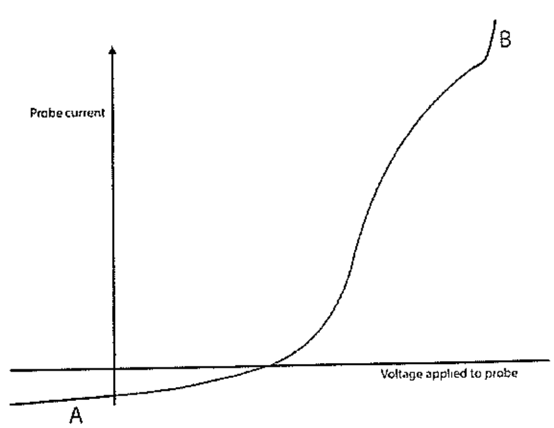

Figure 2

I.4: General Plasma Physics: Weyl particles in magnetized plasmas [35 pts]

Consider the cold plasma waves in a uniform and stationary magnetized plasma. The ions are motionless and the magnetic field is constant, i.e., $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$ ($B_0 > 0$). The linearized fluid equations governing the wave dynamics are

$$\partial_t \boldsymbol{v} = -e\boldsymbol{E}/m_e - \Omega \boldsymbol{v} \times \boldsymbol{e}_z, \tag{1}$$

$$\partial_t \boldsymbol{E} = c\nabla \times \boldsymbol{B} + 4\pi e n_e \boldsymbol{v}, \tag{2}$$

$$\partial_t \boldsymbol{B} = -c\nabla \times \boldsymbol{E}, \tag{3}$$

where $\boldsymbol{v}, \boldsymbol{E}, \boldsymbol{B}$ are perturbed velocity, perturbed electric field, and perturbed magnetic field, $e > 0$ and m_e are the electron charge and mass, c is light speed, $\Omega = eB_0/m_e c$ is the cyclotron frequency, and $n_e > 0$ is the unperturbed electron density. Consider eigenmodes of system in the form of

$$(\boldsymbol{v}, \boldsymbol{E}, \boldsymbol{B}) \sim (\hat{\boldsymbol{v}}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{B}}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t), \tag{4}$$

where $(\hat{\boldsymbol{v}}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{B}})$ are complex vectors and $\boldsymbol{k}$ is the real wavenumber vector. The linear system is 9 dimensional, and thus admits 9 eigenmodes mathematically. It turns out all 9 eigen-frequencies are real. For a given $\boldsymbol{k}$, let's order the eigen-frequencies according to their values as

$$\omega_n(\boldsymbol{k}), n \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}, \tag{5}$$

where $\omega_m(\boldsymbol{k}) \le \omega_n(\boldsymbol{k})$ if $m < n$. It can be shown that $\omega_0(\boldsymbol{k}) = 0$, but it will be ignored because it is unphysical. (That eigenmode would be removed if the condition $\nabla \cdot \boldsymbol{B} = 0$ was included.)

1. [10 points] Prove that the system is stable and undamped, i.e, all eigen-frequencies are indeed real as stated above. [Hint: Show that by normalizing $\boldsymbol{v}$, using a proper combination of n_e and m_e, to the same dimension as the electric field, the governing equations for the eigenmodes can be written as $H|\psi\rangle = \omega|\psi\rangle$, where $|\psi\rangle = (\hat{\boldsymbol{v}}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{B}})^{\mathrm{T}}$ and H is Hermitian. In the Gaussian units, $\boldsymbol{B}$ and $\boldsymbol{E}$ have the same dimension.]

2. [8 points] Prove that for a given $\boldsymbol{k}$ the spectrum is symmetric with respect to the origin on the real axis, i.e., if ω is an eigen-frequency, so is $-\omega$. [Hint: $(\hat{\boldsymbol{v}}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{B}})$ are complex vectors.]

3. [7 points] Name the four eigenmodes $\omega_n(\boldsymbol{k}), n \in \{1, 2, 3, 4\}$ when the waves propagate parallel to $\boldsymbol{B}_0$, i.e., $\boldsymbol{k} = k_\parallel \boldsymbol{e}_z$. Typical dispersion relations for these fours eigenmodes are shown in Fig. 1 on the next page. [Hint: You learned all four modes in detail in AST551-GPPI.]

4. [10 points] For these four modes $\omega_n(\boldsymbol{k}), n \in \{1, 2, 3, 4\}$ when $\boldsymbol{k} = k_\parallel \boldsymbol{e}_z$, show that there exist at least one and at most two wave-wave resonances among them in the open interval of $0 < k_\parallel < \infty$. A wave-wave resonance means that two waves have the same frequency and wavenumber. What is the condition for the existence of two wave-wave resonances as shown in Fig. 1?

   [Weyl particles are mass-less particles predicted theoretically in 1929. Excitations analogous to Weyl particles, known as Weyl quasi-particles, were discovered 85 years later in condensed matter. Plasma waves near the resonances can be viewed as Weyl quasi-particles in plasmas.]

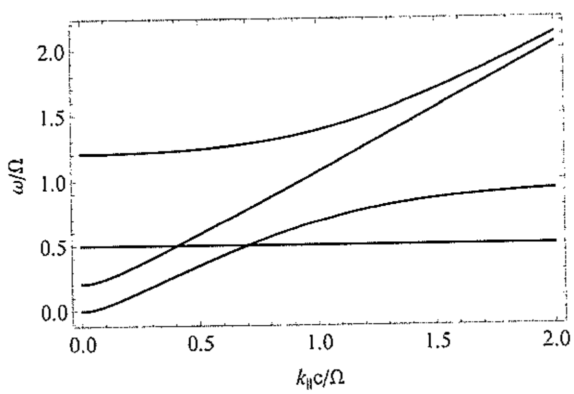

Figure 1: Dispersion relations of the four eigenmodes. The system parameter is $\omega_{\mathrm{p}}/\Omega = 0.5$.

I.5 MHD Short Problem [15 points]

In the adiabatic toroidal compression (ATC) tokamak in Princeton, the vertical field was used to shift the plasma inward in major radius. The major radius after the shift was about half that before. The plasma beta was low. The average toroidal magnetic field in the plasma after the shift was about twice that before the shift.

a. Why would you expect this magnitude of increase in the toroidal field? (3 pts)

In the following questions, assume that the plasma is ideal (zero resistivity). To simplify things, use a large aspect ratio (cylindrical) approximation for the plasma in answering the questions. *(Note that the length of the cylinder representing the shifted plasma is half that of the cylinder representing the unshifted plasma.)* You can also take the shape of the plasma cross-section to be circular.

b. Suppose that the density profile of the unshifted plasma was $n = n_0(1 - r^2/a^2)$, where n_0 is the density at the center of the plasma, r is a radial coordinate that measures the distance from the center of the plasma, and a is the minor radius of the plasma. What was the final density profile after the radial shift? You may assume that the plasma cross-field diffusion was small enough to be neglected in this calculation. (6 pts)

c. Suppose that the toroidal current density profile in the unshifted plasma was $j_t = j_0(1 - r^2/a^2)$. What was the toroidal current density profile after the radial shift? (6 pts)

I.6 General: Particle Orbits [15 points]

Consider particle motion in a magnetic field $\mathbf{B} = \hat{z} B_0 y$ (and $B_0 > 0$). [The sketches below do not need to be precise, but they should illustrate some main qualitative properties.]

(a) [5 points] Sketch the magnetic field in the (x, y) plane. Sketch the orbit of an ion that starts at $y = y_0$, far enough above the midplane ($y = 0$) that a guiding center drift approximation holds. Sketch the orbits, for the same length of time, of ions that starts at $-y_0$ and at $2y_0$.

(b) [5 points] Sketch the orbit of an ion that starts at the midplane with a velocity in the y direction.

(c) [5 points] Sketch the orbits of 3 other ions starting at the midplane, but with a range of possible initial velocity angles to illustrate how the orbit properties depend on the initial angle.

DEPARTMENT OF ASTROPHYSICAL SCIENCES,
PROGRAM IN PLASMA PHYSICS

GENERAL EXAMINATION, PART II

May 9, 2023

9:00 a.m. – 1:00 p.m.

- Today's exam (Part II) contains 6 problems on pages 2–9. Attempt all problems.
- Today's exam has been designed to require three hours of work (180 points). However, you are allowed one extra hour, so the total time allotted for today is four hours. The scores on the questions will be weighted in proportion to their allotted time.
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Contents

Page

II.1 MHD Long Problem (total 40 points)

In this problem, you will be asked to derive the stability condition for ideal MHD interchange instability in magnetically confined plasmas and to apply it to plasmas confined by a current-carrying wire.
Consider two neighboring, narrow magnetic flux tubes (see Figure (1) below): flux tube 1 and flux tube 2, with same magnetic flux $\Phi_1 = \Phi_2 = \Phi$, but different cross-sectional areas S_1 and S_2, different volume V_1 and V_2, and different plasma pressure p_1 and p_2, respectively. Therefore, their magnetic field strengths $B_1 = \Phi/S_1$ and $B_2 = \Phi/S_2$ are different.

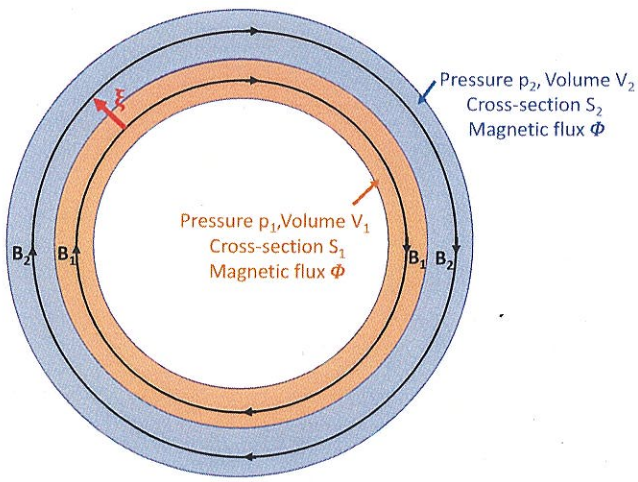


Figure 1: Two flux tubes before interchanging

We consider interchanging the positions of the two flux tubes – *i.e.* moving the plasma in tube 1 into the volume occupied by tube 2 and vice versa. Since we use ideal MHD, the magnetic flux in each tube is preserved during the interchange motions. Heat flow is also neglected so that during the motion the ideal gas law $pV^\gamma = constant$ holds in each tube. The plasma internal energy in a tube is given by $pV/(\gamma - 1)$.

1. [5 points] Show that total magnetic energy of the two flux tubes does not change after the interchange.

2. [15 points] For $p_2 = p_1 + \delta p$ and $V_2 = V_1 + \delta V$ with $\delta p/p_1 \sim \mathcal{O}(\epsilon) \ll 1$ and $\delta V/V_1 \sim \mathcal{O}(\epsilon) \ll 1$, show that the change in the total plasma internal energy (δW_p) of the two flux tubes after the interchange is to $\mathcal{O}(\epsilon^2)$ given by

$$\delta W_p = p_1 \left[\frac{\delta p \delta V}{p_1} + \gamma \frac{(\delta V)^2}{V_1} \right] = p_1 \delta V \delta \left[\ln \left(p V^\gamma \right) \right]. \tag{1}$$

Justify that interchange motion is stable if:

$$\delta W_p > 0. \tag{2}$$

3. Consider the interchange stability condition of Eq. (2) in plasmas confined by an infinitely long, current-carrying wire of current I in the z direction and radius r_0. You may assume that the current carried by the plasma is negligible (this is true if the plasma β is small).

   (a) [5 points] Consider a narrow flux tube at radius $r > r_0$ with flux Φ. Derive an expression for the volume V occupied by a narrow flux tube in terms of Φ, r and I.

   (b) [15 points] Assume that the plasma pressure is $p(r) = p_0 r^{-\alpha}$. Now consider (see Figure (1)) interchanging adjacent flux tubes with radial separation $r_2 = r_1 + \xi$ with $\xi \ll r_1$. Show that $\delta p \sim -\alpha(\xi/r_1)p_1$ and $\delta V \sim 2(\xi/r_1)V_1$. What is the range of α in which the plasma is stable to the interchange instability? What is the profile for plasma β?

II.2 Irreversible Processes [45 pts]

Consider the following 1D–1V linearized Vlasov equation governing the evolution of the distribution function $f(t, z, v) = f_0(v) + \delta f_k(t, v)\exp(\mathrm{i}kz)$, where $f_0 = n\exp(-v^2/v_{\mathrm{th}}^2)/\sqrt{\pi}$ is a stationary, homogeneous, Maxwellian background and δf_k is a small-amplitude perturbation of wavenumber k excited by the source term that appears on the right-hand side:

$$\left(\frac{\partial}{\partial t} + \mathrm{i}kv\right)\delta f_k + \mathrm{i}kv\,\frac{q\varphi_k}{T} f_0 = \frac{2v}{v_{\mathrm{th}}^2}\,a_k(t) f_0. \tag{1}$$

The notation is standard: n is the number density, q is the charge, T is the temperature, and $v_{\mathrm{th}} \doteq (2T/m)^{1/2}$ is the thermal speed for particles of mass m. The electrostatic potential φ_k satisfies

$$\frac{q\varphi_k}{T} = \frac{\alpha}{n}\int_{-\infty}^{\infty} \mathrm{d}v\,\delta f_k, \tag{2}$$

where α is a constant. The acceleration $a_k(t)$ drives velocity fluctuations stochastically at wavenumber k; it is described statistically by its two-time correlation function,

$$\langle a_k(t) a_k^*(t')\rangle = \varepsilon v_{\mathrm{th}}^2\,\delta(t - t') \quad \text{with} \quad \langle a_k(t)\rangle = 0, \tag{3}$$

where ε is a constant. Answer the following.

(a) **[14 points]** Show that the steady-state fluctuation level satisfies

$$\left\langle \left|\frac{q\varphi_k(t)}{T}\right|^2 \right\rangle = \frac{\varepsilon}{|k|v_{\mathrm{th}}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{2\pi}\left|\frac{Z'(\zeta)}{D_\alpha(\zeta)}\right|^2, \tag{4}$$

where $\zeta \doteq \omega/|k|v_{\mathrm{th}}$, $Z(\zeta)$ is the plasma dispersion function, and

$$D_\alpha(\zeta) \doteq \frac{1}{\alpha} + 1 + \zeta Z(\zeta) = \frac{1}{\alpha} - \frac{1}{2}Z'(\zeta)$$

is the dielectric function.

(b) **[7 points]** Solve $D_\alpha(\zeta) = 0$ in the limit $\alpha \gg 1$ to find the approximate dispersion relation for a rapidly oscillating, weakly damped mode:

$$\mathrm{Re}(\zeta) \approx \pm\sqrt{\frac{\alpha}{2}}, \quad \mathrm{Im}(\zeta) \doteq \frac{-\gamma}{|k|v_{\mathrm{th}}} \approx -\sqrt{\pi}\left(\frac{\alpha}{2}\right)^2 \exp\left(-\frac{\alpha}{2}\right). \tag{5}$$

(c) **[10 points]** With $\alpha = k^{-2}(4\pi e^2 n/T) \doteq (k\lambda_{\mathrm{De}})^{-2}$, equation (5) describes long-wavelength ($k\lambda_{\mathrm{De}} \ll 1$) Langmuir oscillations in an electron–ion plasma with unresponsive ions. Substitute (5) into (4) and evaluate the integral to show that the electrostatic energy associated with such Langmuir fluctuations satisfies

$$\left\langle \frac{|E_k|^2}{8\pi n} \right\rangle \approx \frac{T}{2}\frac{\varepsilon}{2\gamma}. \tag{6}$$

(d) **[7 points]** For a stochastically driven, weakly damped, harmonic oscillator in thermal equilibrium, $\varepsilon = 2\gamma$. Use this to interpret (6) physically in the context of the steady-state electrostatic fluctuation level in a weakly coupled plasma in thermal equilibrium. Namely, comment in detail on how such a steady state is achieved, what $a_k(t)$ and ε represent physically in this case, and why the electrostatic energy is equal to $T/2$ at long wavelengths. Finally, how do you expect the steady-state spectrum to scale with k at small wavelengths ($k\lambda_{De} \gg 1$) where Langmuir fluctuations are strongly damped?

(e) **[7 points]** Provide two ways that you would modify $a_k(t)$, ε, and/or α so that equations (1)–(3) provided a more accurate model for how a real plasma generates and interacts with a thermal bath of long-wavelength Langmuir fluctuations. Explain your answer.

Possibly useless information:

$$\lim_{\epsilon\to+0} \frac{1}{x - a \pm \mathrm{i}\epsilon} = \mathrm{PV}\left(\frac{1}{x-a}\right) \mp \mathrm{i}\pi\delta(x-a), \quad \lim_{\epsilon\to+0} \frac{\epsilon}{(x-a)^2+\epsilon^2} = \pi\delta(x-a)$$

$$f(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} f(t), \quad f(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} f(\omega), \quad \delta(\omega) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, \mathrm{e}^{\mathrm{i}\omega t}$$

$$Z(\zeta) \doteq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{e}^{-x^2}}{x-\zeta}, \quad Z'(\zeta) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{x\,\mathrm{e}^{-x^2}}{x-\zeta}, \quad Z'' + 2\zeta Z' + 2Z = 0$$

$$\zeta \ll 1: \; Z(\zeta) \simeq \mathrm{i}\sqrt{\pi}\exp(-\zeta^2) - 2\zeta\left(1 - \frac{2}{3}\zeta^2 - \dots\right)$$

$$|\zeta| \gg 1: \; Z(\zeta) \simeq \mathrm{i}\sqrt{\pi}\sigma\exp(-\zeta^2) - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \dots\right); \; \sigma = \begin{cases} 0 & \mathrm{Im}(\zeta) > |\mathrm{Re}(\zeta)|^{-1} \\ 1 & |\mathrm{Im}(\zeta)| < |\mathrm{Re}(\zeta)|^{-1} \\ 2 & \mathrm{Im}(\zeta) < -|\mathrm{Re}(\zeta)|^{-1} \end{cases}$$

II.3 Applied Math: Resonance and long time asymptotics [45 points]

This question explores the long time asymptotics of solutions of a wave equation with a delta function source:

$$\frac{\partial^2 y}{\partial t^2} = x\frac{\partial}{\partial x}(x\frac{\partial y}{\partial x}) - \omega^2 y + f(t)\delta(x-1). \tag{1}$$

Where the source $f(t) = \sin\omega t$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. We consider solutions on the domain $x > 0$. The initial conditions are:

$$y(t=0,x) = \frac{\partial y}{\partial t}(t=0,x) = 0. \tag{2}$$

You will need the standard Laplace transform relations:

$$\tilde{y}(p,x) = \int_0^\infty dt\, y(t,x)e^{-pt} \quad and \quad y(t,x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{y}(p,x)e^{pt} \tag{3}$$

where the real number σ is chosen such that $\tilde{y}(p,x)$ is analytic for the real part of p greater than σ.

(i) [8 points] We start with an integral that helps you later on. Consider

$$C(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dp}{p^{3/2}}e^{pt} \tag{4}$$

with σ defined as above. Sketch the contour in complex p space showing the non-analytic poles/branchcuts. Find an explicit formula for dC/dt by differentiating this formula and evaluating the resulting integral. Deduce $C(t)$, given this and the initial condition $C(0) = 0$.

(ii) [9 points] Show that $\tilde{f}(p) = \frac{\omega}{p^2+\omega^2}$; and that:

$$(p^2+\omega^2)\tilde{y}(p,x) = x\frac{\partial}{\partial x}(x\frac{\partial \tilde{y}(p,x)}{\partial x}) + \tilde{f}(p)\delta(x-1) \tag{5}$$

(iii) [4 points] By integrating across the delta function source show that

$$\lim_{\epsilon\to 0}\left(\frac{\partial \tilde{y}}{\partial x}(p,1+\epsilon) - \frac{\partial \tilde{y}}{\partial x}(p,1-\epsilon)\right) = -\tilde{f}(p) \tag{6}$$

(iv) [9 points] Derive the solutions

$$y(t,x) = \frac{1}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \frac{\omega}{(p^2+\omega^2)^{3/2}} x^{(p^2+\omega^2)^{1/2}} e^{pt} \qquad for\ \ 0 < x < 1$$

$$y(t,x) = \frac{1}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \frac{\omega}{(p^2+\omega^2)^{3/2}} x^{-(p^2+\omega^2)^{1/2}} e^{pt} \qquad for\ \ 1 < x \tag{7}$$

(v) [13 points] Evaluate $y(t,x)$ for $t\to\infty$ and x finite.

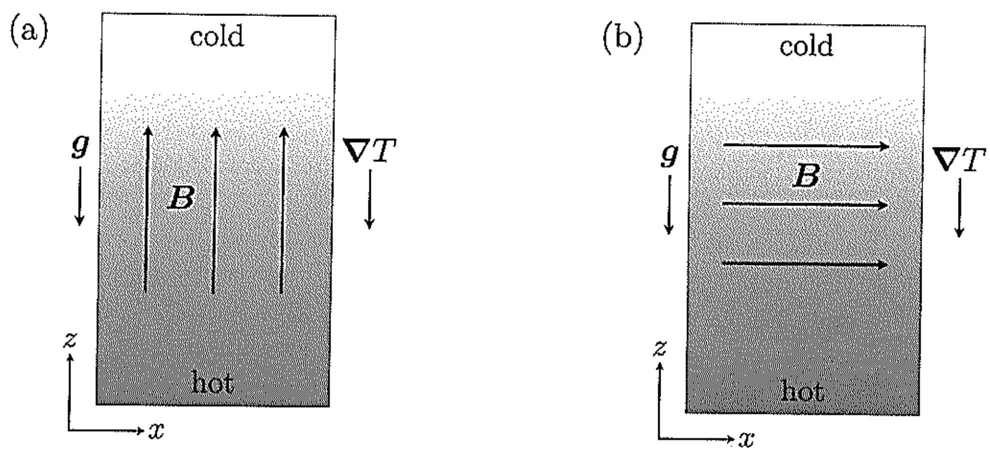
(vi) [2 points] For what values of x does the asymptotic approximation in the previous part break down?

II.4 Transport Theory Short Problem[20 pts]

A collisional, stratified plasma is in hydrostatic equilibrium with a constant gravitational field, such that $\nabla P = -mng\hat{\boldsymbol{z}}$, where P is the thermal pressure, mn is the mass density, and $g > 0$ is a constant. There is a temperature gradient, with hot plasma on the bottom and cold plasma on the top. The plasma is threaded by a constant, uniform magnetic field $\boldsymbol{B}$, whose strength is large enough that the Larmor radii of all particles are much smaller than the collisional mean free paths. Ignoring any possible instabilities, answer the following:

(a) [**9 points**] Suppose $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, as in the figure below. Use physical arguments to estimate the relevant conductive thermal diffusivity. Given such collisional transport, what temperature profile should the atmosphere have to remain in hydrostatic equilibrium?

(b) [**11 points**] Suppose $\boldsymbol{B} = B\hat{\boldsymbol{x}}$, as in the figure below. Use physical arguments to estimate the relevant conductive thermal diffusivity. Given such collisional transport, what temperature profile should the atmosphere have to remain in hydrostatic equilibrium?



II.2 Waves Short Problem: Plasma heating on a density gradient [20 points]

Consider a cold nonmagnetized collisionless plasma with inhomogeneous background electron density n_0. Assume that ions form a neutralizing background and remain stationary.

(a) [**7 points**] Show that the linear charge-density perturbation $\tilde{\rho}$ satisfies

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} + \omega_p^2 \tilde{\rho} = -\frac{e^2}{m} \nabla n_0 \cdot \tilde{\boldsymbol{E}}, \tag{1}$$

where ω_p is the local plasma frequency, e and m are the electron charge and mass, respectively, and $\tilde{\boldsymbol{E}}$ is the electric field.

(b) [**4 points**] Assume $n_0 = n_c e^{x/L}$ with constant n_c and L. Using (1), explain how a stationary electromagnetic wave incident from vacuum ($x \to -\infty$) can heat such a plasma. For a given wave frequency ω, at what x can the heating occur? Also, what should be the wave polarization for the heating to occur?

(c) [**9 points**] Assuming the wavevector $\boldsymbol{k} = (k_x, k_y, 0)$, qualitatively analyze how the amount of power deposited depends on the angle of incidence $\theta = \arctan(k_y/k_x)|_{x \to -\infty}$.

Hint: Consider the fact that the wave amplitude rapidly, yet not instantaneously, decreases to zero beyond the geometrical-optics cutoff. Also consider the wave polarization.

II.6 Experimental methods short problem [10 pts]

For each, a few sentences should be sufficient. Formulas could help you but are not required. No essays.

a. (5 pts) Describe one diagnostic for measuring plasma conditions in ITER ($n_e \sim 10^{20}$ m^{-3}, $T_e \sim T_i \sim 15$ keV). You can choose density or temperature. Describe the physics principle of the measurement, and briefly what hardware is suitable as a detector.

b. (5 pts) Describe one diagnostic for measuring plasma conditions in a laser-heated high energy density plasma, at near-to-above solid density conditions ($n_e \sim 10^{29}$ m^{-3}, $T_e \sim T_i \sim$ 1 keV). You can choose density or temperature. Describe the physics principle of the measurement, and the hardware that is needed as a detector.