2022 II.3

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1 Part a

Consider the sheared magnetic field in equilibrium

$$\mathbf{B} = b\tanh\left(\frac{x}{L}\right)\hat{\mathbf{y}} + B_0\hat{\mathbf{z}} \tag{1}$$

We are asked to represent this field in the following form by obtaining the flux function $\Psi_0(x, y)$

$$\mathbf{B} = \hat{z} \times \nabla\Psi_0(x, y) + B_0\hat{\mathbf{z}} \tag{2}$$

Since there is no $\hat{x}$ component of $\mathbf{B}$, $\nabla\Psi_0$ is only in the $\hat{x}$ direction and $\Psi_0(x, y) = \Psi_0(x)$.

$$\frac{d\Psi_0}{dx} = b\tanh\left(\frac{x}{L}\right) \tag{3}$$

$$\Psi_0 = bL\log\left(\cosh\left(\frac{x}{L}\right)\right) \tag{4}$$

You can do the integration with a u-sub: $\tanh(x) = \sinh(x)/\cosh(x)$, $u = \cosh(x), \partial_x u = \sinh(x)$
The other task is to find $\mathbf{J_0} = \frac{1}{\mu_0}\nabla \times \mathbf{B}$

$$\mathbf{J_0} = \frac{b}{\mu_0 L}\operatorname{sech}^2\left(\frac{x}{L}\right)\hat{z} \tag{5}$$

2 Part b

The resistive Ohm's Law is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta\mathbf{J} \tag{6}$$

Although it's tempting to use the induction equation instead of Ohm's Law, they explicitly ask to derive from Ohm's Law so this is our starting point. We can express $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$. We assume quasineutrality in MHD so there are no electrostatic gradients and we can drop the second term: $\mathbf{E} = -\partial\mathbf{A}/\partial t$. Then the goal is to express $\mathbf{A}_1$ in terms of Ψ_1.

$$\mathbf{B_1} = \nabla \times \mathbf{A_1} = \hat{z} \times \nabla\Psi_1(x, y) \tag{7}$$

There may be a more efficient way to see this, but after writing out the components of these equations we can see that $\mathbf{A_1} = \Psi_1\hat{\mathbf{z}}$.

Therefore, we can sub this into Ohm's Law in the $\hat{\mathbf{z}}$ direction.

$$-\frac{\partial\Psi_1}{\partial t} + v_0 \times B_1 + v_1 \times B_0 + v_1 \times B_1 = \eta J_1 \tag{8}$$

Here, we've taken the 0th order solution to cancel. The nonlinear term is $v_1 \times B_1$. The Lundquist number is defined to be the ratio of the resistive timescale to Alfven Timescale $S = \tau_R/\tau_A = \mu_0 v_A L/\eta$. Note: this can be rederived from twiddle algebra in the induction equation by considering only the convective term on the RHS to get τ_A and only the diffusive term on the RHS to get τ_R. This number appears on Sweet Parker problems too so it's worth knowing how to derive it quickly.

After solving for η in terms of S and subbing in, we get:

$$\frac{\partial \Psi_1}{\partial t} = v_0 \times B_1 + v_1 \times B_0 + v_1 \times B_1 - \frac{\mu_0 v_A L}{S} J_1 \tag{9}$$

where B_1 and J_1 are defined

$$B_1 = \hat{z} \times \nabla \Psi_1 \tag{10}$$

$$J_1 = \frac{1}{\mu_0} \nabla \times B_1 \tag{11}$$

3 Part c

In the "constant-Ψ" approximation, the flux function is constant along the island boundary.

$$\Psi = \Psi_0 + \Psi_1 = \text{Constant we can choose} = 0 \tag{12}$$

$$bL \log\left(\cosh\left(\frac{x}{L}\right)\right) + \Psi_1(t)\cos(ky) = 0 \tag{13}$$

Taylor expand Ψ_0 for small x using $\cosh(\epsilon) \approx 1 + \epsilon^2/2$, $\log\left(1 + \epsilon^2/2\right) \approx \epsilon^2/2$ and take the island width W equal to $2x$.

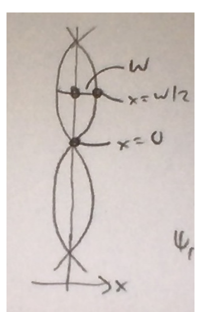
$$bL\frac{x^2}{2L^2} = -\Psi_1(t)\cos(ky) \tag{14}$$

$$b\frac{W^2}{8L} = -\Psi_1(t)\cos(ky) \tag{15}$$

$$W(t) = \sqrt{\frac{8L\Psi_1(t)\cos(ky)}{b}} \tag{16}$$

$$\max(W(t)) = \sqrt{\frac{8L\Psi_1(t)}{b}} \tag{17}$$

In the limit of ideal MHD, the magnetic field is frozen into the fluid and the magnetic topology cannot change. Assuming an instantaneous change from resistive to ideal MHD, the island width growth is halted. Therefore, the island width is a constant value at whatever it was before this change.



4 Part d

In the linear regime, the island experiences a growth rate $\gamma_L \sim S^{-3/5}/\tau_A$ according to the tearing mode growth rate. However, the growth saturates and slows to $\gamma_{NL} \sim 1/\tau_R$ in the nonlinear regime when the island width becomes comparable to the resistive layer. At this point, there are significant modifications to the underlying magnetic structure (see Goldston page 360 or Bhattacharjee page 269 for more info).