## 2022 I. 5

Jesse Griff-McMahon
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Given equation:

$$
\begin{equation*}
x=\exp (x)-\alpha \tag{1}
\end{equation*}
$$

## 1 Part a

Draw three curves for $y=\exp (x)-\alpha$ using $\alpha=0,1,2$. Also draw line $y=x$.
The lines $y=x$ and $y=\exp (x)-\alpha$ intersect only when $\alpha \geq 1$.
When $\alpha=1$, there is one solution at $x=0$ because the curves are tangent (same slope and y value). When $\alpha>1$, there are two solutions for positive and negative x .


## 2 Part b

$$
x_{n+1}=e^{x_{n}}-\alpha \quad \text { for } \quad \alpha>1
$$




## 3 Part c

We can consider an iteration scheme that solves for x on the RHS of the given equation.

$$
\begin{array}{r}
x=\exp (x)-\alpha, \\
\log (x+\alpha)=x, \\
x_{n+1}=\log \left(x_{n}+\alpha\right) \tag{4}
\end{array}
$$

We can show, graphically, that this new scheme converges to the positive solution when $x_{0}$ is greater than the negative solution.


To show whether the convergence is linear or quadratic, we assume there is an iterated solution $x_{n}$ that is very close to the actual solution $\hat{x}$

$$
\begin{equation*}
x_{n}=\hat{x}+\Delta_{n} \tag{5}
\end{equation*}
$$

where $\Delta_{n}$ is small.

The next iteration, $x_{n+1}=\hat{x}+\Delta_{n+1}$, can be calculated using the iterative scheme:

$$
\begin{array}{r}
\hat{x}+\Delta_{n+1}=\log \left[\hat{x}+\alpha+\Delta_{n}\right] \\
\hat{x}+\Delta_{n+1}=\log \left[(\hat{x}+\alpha)\left(1+\frac{\Delta_{n}}{\hat{x}+\alpha}\right]\right. \\
\hat{x}+\Delta_{n+1}=\log [\hat{x}+\alpha]+\log \left[1+\frac{\Delta_{n}}{\hat{x}+\alpha}\right] \tag{8}
\end{array}
$$

The 0th order solution cancels and we taylor expand $\log (1+\epsilon) \approx \epsilon$ to yield the asymptotic convergence relation.

$$
\begin{equation*}
\Delta_{n+1}=\frac{\Delta_{n}}{\hat{x}+\alpha} \tag{9}
\end{equation*}
$$

This is a linear convergence because $\Delta_{n} \rightarrow 0$ proportionally to $\Delta_{n}$.

