2018 II.1

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1 Part A (5 points)

We are going to perturb the magnetic field energy $W = \int (B^2/2\mu_0)dV$ and set the perturbed magnetic field energy to 0. This tells us that we are at a minimum of the magnetic field energy. If the plasma beta is much less than 1, then the magnetic field energy approximates the total energy (remember: $W = \rho u^2/2 + P/(\gamma - 1) + B^2/2\mu_0$), and we are at a minimum of total energy if the first-order variation in the energy is zero.

The variation in the magnetic field energy is

$$\delta W = \frac{1}{2\mu_0}\delta \int \boldsymbol{B} \cdot \boldsymbol{B} dV = \frac{1}{\mu_0} \int \boldsymbol{B} \cdot \boldsymbol{B}_1 dV \tag{1}$$

Now, from the ideal MHD induction equation we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \tag{2}$$

or using $\boldsymbol{u} = \frac{\partial \boldsymbol{\xi}}{\partial t}$, we have

$$\boldsymbol{B}_1 = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}) \tag{3}$$

which tells us that $\boldsymbol{A}_1 = \boldsymbol{\xi} \times \boldsymbol{B}$. So plugging in equation (3) to equation (1) gives us

$$\delta W = \frac{1}{\mu_0} \int \boldsymbol{B} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}) dV \tag{4}$$

We then use the vector identity given to us as a hint, to write our perturbed magnetic energy as

$$\delta W = \frac{1}{\mu_0}\left[\int \boldsymbol{\nabla} \cdot [(\boldsymbol{\xi} \times \boldsymbol{B}) \times \boldsymbol{B}] dV + \int (\boldsymbol{\xi} \times \boldsymbol{B}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) dV \right] \tag{5}$$

We assume that $\boldsymbol{\xi}$ vanishes on the boundary, which kills the divergence term. We can use the second identity in the hint to rewrite the second term, which gives us

$$\delta W = \frac{1}{\mu_0} \int \boldsymbol{\xi} \cdot [\boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{B})] dV = 0 \tag{6}$$

Therefore, $\boldsymbol{B}$ is parallel to $\boldsymbol{\nabla} \times \boldsymbol{B}$, and therefore we have

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha(\boldsymbol{r})\boldsymbol{B} \tag{7}$$

where α is a scalar function. Note that α is not necessarily constant.

2 Part B (10 points)

Now we're going to perturb the magnetic energy, such that the magnetic helicity is held constant. We're going to find that we also get the condition for a force-free field, that $\boldsymbol{B} \parallel \boldsymbol{\nabla} \times \boldsymbol{B}$, except we're going to find that the constant between $\boldsymbol{B}$ and $\boldsymbol{\nabla} \times \boldsymbol{B}$ is a scalar rather than a scalar function.

When we want to minimize a function subject to some constraint, we use a Lagrange multiplier times the function which is set to 0, in this case the magnetic helicity H minus some constant C. Therefore, our Lagrangian (function to minimize) is

$$\mathcal{L} = W + \mu(H - C) = \frac{1}{2\mu_0}\left[\int B^2 dV - \mu\Big(\int \boldsymbol{A} \cdot \boldsymbol{B} dV - C\Big)\right] \tag{8}$$

We'll take the variation in $\mathcal{L}$ and set it to zero. This gives us

$$\delta\mathcal{L} = 0 = \frac{1}{2\mu_0}\left[\int 2\boldsymbol{B} \cdot \boldsymbol{B}_1 dV - \mu \int \boldsymbol{B} \cdot \boldsymbol{A}_1 + \boldsymbol{A} \cdot \boldsymbol{B}_1 dV\right] \tag{9}$$

$$0 = \int \Big[2\boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}_1) - \mu\boldsymbol{B} \cdot \boldsymbol{A}_1 - \mu\boldsymbol{A} \cdot \boldsymbol{\nabla} \times \boldsymbol{A}_1\Big] dV$$

Using the same identity twice, we have

$$0 = \int \boldsymbol{\nabla} \cdot (2\boldsymbol{A}_1 \times \boldsymbol{B} + \mu\boldsymbol{A} \times \boldsymbol{A}_1) dV + \int \boldsymbol{A}_1 \cdot 2(\boldsymbol{\nabla} \times \boldsymbol{B}) - 2\mu\boldsymbol{A}_1 \cdot \boldsymbol{\nabla} \times \boldsymbol{A} dV$$

Assuming that $\boldsymbol{A}_1$ vanishes on the boundary, the divergence term goes to 0. The second term is then

$$0 = 2\int \boldsymbol{A}_1 \cdot (\boldsymbol{\nabla} \times \boldsymbol{B} - \mu\boldsymbol{B}) dV \tag{10}$$

which tells us that

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu\boldsymbol{B} \tag{11}$$

where μ is a constant equal to the Lagrange multiplier of the system. This is also a force-free condition.

3 Part c (5 points)

In both (a) and (b), the current is proportional to the magnetic field. The difference is that in (b), the current and the magnetic field are related by a constant, while in (a) the constant is a scalar function which can vary in space.

4 Part d (5 points)

The first, second, and third terms are all a positive constant times a squared quantity. These are all positive semi-definite. The fourth term is zero because by the MHD equilibrium equation, $\boldsymbol{J}\times\boldsymbol{B} = \boldsymbol{\nabla} P = 0$, so $\boldsymbol{\nabla} P = 0$. The fifth term is either positive or negative. Therefore, in part (e) we will evaluate whether the fifth term is strictly greater than or equal to 0 or can be less than zero. If it can ever be less than zero, then by the energy principle our plasma is unstable.

5 Part e (10 points)

Let's evaluate the fifth term to see whether it can ever be less than zero.

We see that $\boldsymbol{\xi}_\perp \times \boldsymbol{B} = \boldsymbol{A}_1$, and $Q_\perp = \boldsymbol{B}_{1\perp}$. Therefore, the fifth term equals

$$-\frac{\boldsymbol{J}\cdot\boldsymbol{B}}{B^2}\int dV\,\boldsymbol{A}_{1\perp}\cdot\boldsymbol{B}_{1\perp} \tag{12}$$

Now, since $\boldsymbol{A}_1 \perp \boldsymbol{B}$, then $\boldsymbol{A}_1 = \boldsymbol{A}_{1\perp}$, so $\boldsymbol{A}_{1\perp}\cdot\boldsymbol{B}_{1\perp} = \boldsymbol{A}_1\cdot\boldsymbol{B}_1$. So our last term in the energy integral becomes

$$-\frac{\boldsymbol{J}\cdot\boldsymbol{B}}{B^2}\int dV\,\boldsymbol{A}_1\cdot\boldsymbol{B}_1 \tag{13}$$

For part (a), the magnetic helicity is not conserved, so this quantity can be less than zero, and our plasma can be unstable.

For part (b), we have magnetic helicity is exactly conserved, so the integral is automatically zero. Therefore, a plasma which conserves helicity and relaxes to a minimum energy state is stable.