4) MHD - Force Free Flux Tube

We consider a force free flux tube. This means $J \times B = \nabla P = 0$. It is sometimes called a parallel pinch because $J \parallel B$. [1] This is actually a constraint that

$$P(r) = P_0$$

We are also given "an equal number radians is twisted per unit length". Let us interpret this to mean axial length $d\theta/dz = k$ is constant. This is the same as specifying the safety factor

$$q(r) = q_0$$

Together, these two functions uniquely specify an MHD system. This system is called the Gold-Hoyle flux tube.[2] It happens to be a 1D system (all variables depend only on r) which lives in 3 dimensions.

a) magnetic fields

We can use Ampere's law to write

$$\mu_0 J = \nabla \times B = \frac{1}{r}\begin{pmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & 0 & 0 \\ 0 & rB_\theta & B_z \end{pmatrix} = \begin{pmatrix} 0 \\ -\partial_r B_z \\ \frac{1}{r}\partial_r(rB_\theta) \end{pmatrix}$$

Then the force-free condition gives

$$0 = J \times B = -\frac{1}{\mu_0}\left[\frac{1}{r}B_\theta \partial_r(rB_\theta) + B_z \partial_r B_z\right]$$

which implies

$$B_z \partial_r B_z = -\frac{1}{r} B_\theta \partial_r (rB_\theta)$$

For the second constraint we can write the field line following equation

$$\frac{d\vec{r}}{d\vec{B}} = \frac{r d\theta}{B_\theta} = \frac{dz}{B_z}$$

which yields

$$k = \frac{d\theta}{dz} = \frac{B_\theta}{rB_z}$$

So we have two variables and two equations. Let us eliminate $B_\theta = krB_z$ and solve

$$\partial_r B_z = -\partial_r\left[(kr)^2 B_z\right]$$

We can solve this first order ODE

$$\frac{B_z'}{B_z} = -\frac{2k^2 r}{1+k^2r^2} = -\frac{d}{dr}\ln(1+k^2r^2)$$

[1] See Freidberg, *Ideal MHD* Ch 5

[2] T. Gold, F. Hoyle, "On the origin of solar flares," *Royal Astronomical Society* (1960) link.

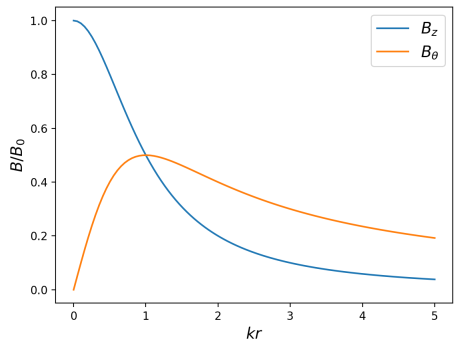
and use the convention $B(r=0) = B_0$ on axis to write

$$B_z(r) = B_0 \left(\frac{1}{1+k^2r^2} \right)$$

It follows immediately that

$$B_\theta(r) = B_0 \left(\frac{kr}{1+k^2r^2} \right)$$

You can check that this satisfies $d\theta/dz = B_\theta/rB_z$.



Note that $B_\theta = B_z$ at $kr = 1$, where B_θ attains a max, and $B_\theta > B_z$ for $kr > 1$.

b) flux and force balance

Now we are told the flux tube has some finite radius a. Let us calculate the axial flux

$$\Phi = \int B \cdot dA = \pi B_0 \int_0^a \frac{2r}{1+k^2r^2} dr = \pi a^2 B_0 \left[\frac{\ln(1+k^2a^2)}{k^2a^2} \right]$$

Next we are asked, what if there is kinetic gas pressure P outside the flux tube? The flux tube defines a region where $\vec{J}$ flows such that $J \times B = 0$. Outside the flux tube $\vec{J} = 0$. This means $B_z(r > a) = 0$, but the poloidal field still exists from the current enclosed within the flux tube

$$B_\theta(r > a) = B_\theta(a)\frac{a}{r}$$

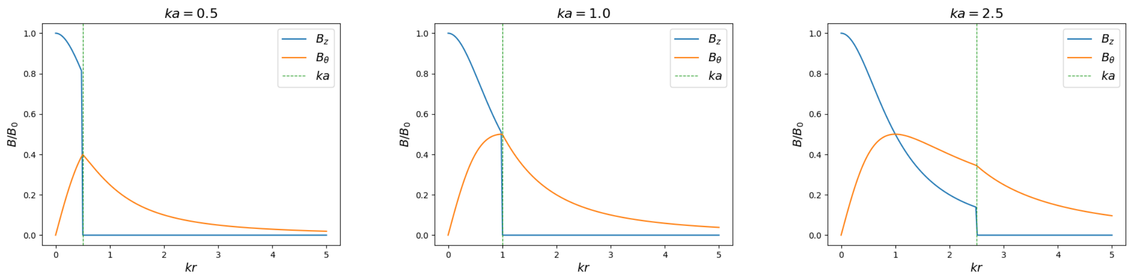
Since B_θ is continuous, both sources of poloidal pressure are continuous across the flux tube boundary. [3] That means the pressure drop is maintained by the axial field along

$$\Delta P = \frac{B_z^2(a)}{2\mu_0} = \frac{B_0^2}{2\mu_0} \left(\frac{1}{1+k^2a^2} \right)^2$$

[3] In general force balance in screw pinch is described by

$$\frac{d}{dr} \left(P + \frac{B_z^2 + B_\theta^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

For our case (the parallel pinch) $J \times B = \nabla P = 0$ so the first term vanishes.

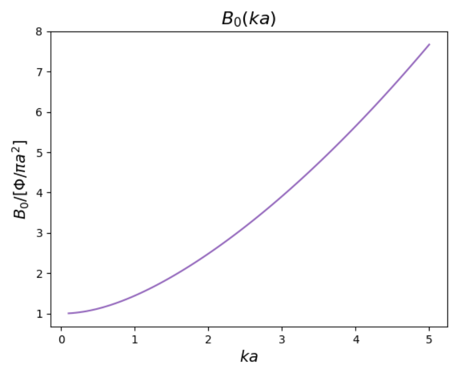


c) increasing twist

What happens if we increase the twist $k = d\theta/dz$ while hold Φ constant? Flux is linear in axial field B_0 so this yields a straightforward relation

$$B_0 = \left(\frac{\Phi}{\pi a^2}\right)\frac{k^2a^2}{\ln(1+k^2a^2)}$$

Given constant flux, B_0 increases monotonically with ka.



What if we also hold constant the pressure outside the flux tube?

$$B_0 = (1+k^2a^2)\sqrt{2\mu_0\Delta P}$$

This appears to contradict the constant flux constraint, unless $\Delta P = P - P_0$ for some finite internal pressure P_0. Previously the internal pressure was a free function which may have varied against B_0. [4] Now with the additional constraint that both Φ and P external be constant (while a is also implicitly held constant) we can specify

$$P_0 = P - \frac{B_0^2}{2\mu_0}\left(\frac{1}{1+k^2a^2}\right)^2 = P - \frac{1}{2\mu_0}\left(\frac{k^2a^2}{1+k^2a^2}\right)^2\left[\frac{\Phi/\pi a^2}{\ln(1+k^2a^2)}\right]^2$$

[4] Indeed we see in the general screw pinch (previous footnote) pressure P and axial field $B_z^2/2\mu_0$ are interchangeable (!), up to boundary conditions.