## 1 Dispersing Beam

We consider a distribution that starts from a monoenergetic beam

$$
f(t=0, \mathbf{v})=\frac{n}{2 \pi v_{0}^{2}} \delta\left(v-v_{0}\right) \delta(\xi-1)
$$

where $\xi=v_{\|} / v=\cos \theta$ is the pitch angle relative to the initial condition. Let us calls this initial distribution $f_{0}$. It evolves under a Lorentz collision operator

$$
\begin{aligned}
C[f] & =\frac{\nu}{2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial f}{\partial \theta}\right) \\
& =\frac{\nu}{2} \frac{\partial}{\partial \xi}\left[\left(1-\xi^{2}\right) \frac{\partial f}{\partial \xi}\right] \\
& =\nu L[f]
\end{aligned}
$$

These are " 3 forms" of the same expression which might come in hand at different points of the problem.

## a) short time diffusion

For short times $(\nu t \ll 1)$ the beam has only diffused slightly. We would like to use this to find an approximate form for $f$. The small angle approximation comes to mind

$$
\sin \theta \approx \theta
$$

This makes the first form of $C[f]$ look like a 2D cylindrical Laplacian

$$
C[f] \approx \frac{\nu}{2}\left[\frac{1}{\theta} \frac{\partial}{\partial \theta}\left(\theta \frac{\partial f}{\partial \theta}\right)\right]
$$

where $\theta$ plays the role of radius. ${ }^{1}$ As a result we may write

$$
\frac{d f}{d t}=C[f]
$$

as

$$
\partial_{t} f-\frac{\nu}{2} \nabla^{2} f=f_{0} \delta(t)
$$

Using the Green's function method, we can write

$$
G\binom{t, R}{t^{\prime}, R^{\prime}}=\frac{1}{4 \pi \tau} e^{-\left(R-R^{\prime}\right)^{2} / 4 \tau}
$$

[^0]for $\tau=t-t^{\prime}>0$, as prescribed in the problem statement. In our case, this translates to
$$
G\binom{t, \theta}{t^{\prime}, \theta^{\prime}}=\frac{1}{2 \pi \nu \tau} e^{-\left(\theta-\theta^{\prime}\right)^{2} / 2 \nu \tau}
$$

Then taking a 2D convolution with the source yields

$$
\begin{aligned}
f(\nu t \ll 1, v, \theta) & =\int G\binom{t, \theta}{0, \theta^{\prime}} f_{0}\left(\theta^{\prime}\right) d \theta^{\prime} d \phi^{\prime} \\
& =\frac{n}{2 \pi v_{0}^{2}} \delta\left(v-v_{0}\right) \int \frac{e^{-\left(\theta-\theta^{\prime}\right)^{2} / 2 \nu \tau}}{\nu \tau} \delta\left(\theta^{\prime}\right) d \theta^{\prime} \\
& =\frac{n}{2 \pi v_{0}^{2}} \delta\left(v-v_{0}\right) \frac{e^{-\theta^{2} / 2 \nu \tau}}{\nu \tau}
\end{aligned}
$$

The initial average change in $\theta$ can be described ${ }^{2}$

$$
\left\langle\theta^{2}\right\rangle=\frac{\int \theta^{2} f(\theta) d \theta}{\int f(\theta) d \theta}=2 \nu \tau
$$

## b) long time diffusion

Now we would like to consider later times when $\left\langle\theta^{2}\right\rangle \sim 1$. Here it is more useful to use the 3 rd form of $C[f]=\nu L[f]$ where we know that Legendre polynomials $P_{l}(\xi)$ are the eigenfunctions

$$
L[f]=\sum_{l=0}^{\infty} L\left[P_{l}(\xi)\right]=-\sum_{l=0}^{\infty} \frac{l(l+1)}{2} P_{l}(\xi)
$$

Since $C[f]$ only has pitch-angle scattering, and no slowing down or $\nu(v)$, the beam stays monoenergetic. It follows that in the Legendre basis

$$
\frac{\partial P_{l}}{\partial t}=-\frac{\nu}{2} l(l+1) P_{l}
$$

Then

$$
f_{l}(t, \xi)=f_{l, 0} e^{-\frac{\nu t}{2} l(l+1)} P_{l}(\xi)
$$

Since the Legendre polynomial orthonormality condition is

$$
\int_{-1}^{1} P_{l}(\xi) P_{m}(\xi) d \xi=\frac{\delta_{l m}}{l+\frac{1}{2}}
$$

We have

$$
\begin{equation*}
f(t, v, \xi)=\frac{n}{2 \pi v_{0}^{2}} \delta\left(v-v_{0}\right) \sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right) P_{l}(\xi) e^{-\frac{\nu t}{2} l(l+1)} \tag{1}
\end{equation*}
$$

In the limit $\nu t \rightarrow \infty$ we see that all terms exponentially damp, except for $P_{0}(\xi)=1$. This makes sense because the Lorentz operator tends toward isotropy.

[^1]
## c) Fokker-Planck coefficients

We are asked to calculate a few more expactation values using Eq (1). Let us first recognize

$$
\int f(t, v, \xi) d \xi d v d \phi=n
$$

Now we may compute
1.

$$
\left\langle v_{\|}\right\rangle=\frac{\int(v \xi) f(t, v, \xi) d \xi d v d \phi}{\int f(t, v, \xi) d \xi d v d \phi}=v_{0} e^{-\nu t}
$$

2. 

$$
\begin{aligned}
\left\langle v_{\|}^{2}\right\rangle & =\frac{\int(v \xi)^{2} f(t, v, \xi) d \xi d v d \phi}{\int f(t, v, \xi) d \xi d v} \\
& =v_{0}^{2} \int\left[\frac{2}{3} P_{2}(\xi)+\frac{1}{3} P_{0}(\xi)\right] \sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right) P_{l}(\xi) e^{-\frac{\nu t}{2} l(l+1)} d \xi \\
& =\frac{v_{0}^{2}}{3}\left(1+2 e^{-3 \nu t}\right)
\end{aligned}
$$

3. 

$$
\left\langle v_{\perp}^{2}\right\rangle=\left\langle v^{2}-v_{\|}^{2}\right\rangle=v_{0}^{2}-\left\langle v_{\|}^{2}\right\rangle=\frac{2 v_{0}^{2}}{3}\left(1-e^{-3 \nu t}\right)
$$

We can see that the distribution starts all $\left\langle v_{\|}^{2}\right\rangle$ and none $\left\langle v_{\perp}^{2}\right\rangle$ but then asymptotes to a $1 / 3: 2 / 3$ mix, which agrees with isotropy. To compute the Fokker Planck coefficients we should consider the "jump moments"

$$
\begin{aligned}
\mathbf{A}(\mathbf{v}) & =\lim _{\Delta t \rightarrow 0}\left\langle\frac{\Delta \mathbf{v}}{\Delta t}\right\rangle \\
\mathbf{B}(\mathbf{v}) & =\lim _{\Delta t \rightarrow 0}\left\langle\frac{\Delta \mathbf{v} \Delta \mathbf{v}}{\Delta t}\right\rangle
\end{aligned}
$$

where

$$
\Delta \mathbf{v}=\mathbf{v}(t)-\mathbf{v}_{0}
$$

Let us start with the frictional drag vector

$$
\begin{aligned}
\mathbf{A}(\mathbf{v}) & =\lim _{\Delta t \rightarrow 0}\left\langle\frac{\Delta \mathbf{v}}{\Delta t}\right\rangle \\
& =\lim _{\Delta t \rightarrow 0} \mathbf{v}_{0}\left(\frac{e^{-\nu \Delta t}-1}{\Delta t}\right) \\
& =-\nu \mathbf{v}_{0}
\end{aligned}
$$

Notice that $\left\langle\mathbf{v}_{\perp}\right\rangle=0$ for all time. So $\langle\mathbf{v}(t)\rangle$ stays parallel to $\mathbf{v}_{0}$, and decreases in magnitude until eventually there is no directed motion on average. Next let us consider

$$
\begin{aligned}
\langle\Delta \mathbf{v} \Delta \mathbf{v}\rangle & =\left(\begin{array}{cc}
\left\langle\Delta \mathbf{v}_{\|} \Delta \mathbf{v}_{\|}\right\rangle & \left\langle\Delta \mathbf{v}_{\|} \Delta \mathbf{v}_{\perp}\right\rangle \\
\left\langle\Delta \mathbf{v}_{\|} \Delta \mathbf{v}_{\perp}\right\rangle & \left\langle\Delta \mathbf{v}_{\perp} \Delta \mathbf{v}_{\perp}\right\rangle
\end{array}\right) \\
& =\left(\begin{array}{cc}
\left\langle\Delta \mathbf{v}_{\|} \Delta \mathbf{v}_{\|}\right\rangle & 0 \\
0 & \left\langle\Delta \mathbf{v}_{\perp} \Delta \mathbf{v}_{\perp}\right\rangle
\end{array}\right) \\
& =\left(\begin{array}{cc}
\left\langle\left(\mathbf{v}_{\|}-\mathbf{v}_{0}\right)\left(\mathbf{v}_{\|}-\mathbf{v}_{0}\right)\right\rangle & 0 \\
0 & \left\langle\mathbf{v}_{\perp} \mathbf{v}_{\perp}\right\rangle
\end{array}\right) \\
& =\mathbf{v}_{0} \mathbf{v}_{0}\left(\begin{array}{cc}
1-2 \frac{\left\langle v_{\|}\right\rangle}{v_{0}}+\frac{\left\langle v_{\|}^{2}\right\rangle}{v_{0}^{2}}
\end{array}\right)+\left(v_{0}^{2} \mathbf{I}-\mathbf{v}_{0} \mathbf{v}_{0}\right) \frac{\left\langle v_{\perp}^{2}\right\rangle}{v_{0}^{2}} \\
& =\mathbf{v}_{0} \mathbf{v}_{0} \frac{2}{3}\left[1-3 e^{-\nu \Delta t}+2 e^{-3 \nu \Delta t}\right]+\mathbf{I} \frac{2}{3} v_{0}^{2}\left(1-e^{-3 \nu \Delta t}\right) \\
& =\frac{2}{3} v_{0}^{2}\left(\begin{array}{cc}
2-3 e^{-\nu \Delta t}+e^{-3 \nu \Delta t} & 0 \\
0 & 1-e^{3 \nu \Delta t}
\end{array}\right)
\end{aligned}
$$

Then the velocity diffusion tensor is

$$
\begin{aligned}
\mathbf{B}(\mathbf{v}) & =\lim _{\Delta t \rightarrow 0}\left\langle\frac{\Delta \mathbf{v} \Delta \mathbf{v}}{\Delta t}\right\rangle \\
& =\lim _{\Delta t \rightarrow 0} \frac{2}{3} \frac{v_{0}^{2}}{\Delta t}\left(\begin{array}{cc}
2-3 e^{-\nu \Delta t}+e^{-3 \nu \Delta t} & 0 \\
0 & 1-e^{3 \nu \Delta t}
\end{array}\right) \\
& =\frac{2}{3} v_{0}^{2}\left(\begin{array}{cc}
-3 \nu+3 \nu & 0 \\
0 & 3 \nu
\end{array}\right) \\
& =2 \nu v_{0}^{2}\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right) \\
& =2 \nu\left(v_{0}^{2} \mathbf{I}-\mathbf{v}_{0} \mathbf{v}_{0}\right)
\end{aligned}
$$

In conclusion

$$
\begin{gathered}
\mathbf{A}(\mathbf{v})=-\nu\binom{v_{0}}{0} \\
\mathbf{B}(\mathbf{v})=2 \nu\left(\begin{array}{cc}
0 & 0 \\
0 & v_{0}^{2}
\end{array}\right)
\end{gathered}
$$

This means particles drag at the collision rate $\nu$, and they diffuse at the random walk rate $2 \nu$.

## 2 Braginskii Transport

We are given the 0th order Maxwellian

$$
f_{0}(t, \mathbf{r}, v)=\frac{n}{\pi^{3 / 2} v_{T}^{3}} e^{-\left(w / v_{T}\right)^{2}}
$$

where $v_{T}^{2}=2 T(t, \mathbf{r}) / m$. We are also given the first order equation

$$
\begin{equation*}
\Omega \frac{\partial f_{1}}{\partial \theta}+C\left[f_{1}\right]=f_{0}\left[\left(\frac{w^{2}}{v_{T}^{2}}-\frac{5}{2}\right) \mathbf{w} \cdot \nabla \ln T+\frac{2}{v_{T}^{2}} \mathbf{w} \mathbf{w}: \nabla \mathbf{u}\right] \tag{2}
\end{equation*}
$$

## d) parallel viscous stress

We take the $C\left[f_{1}\right]=\nu L\left[v_{1}\right]$ to be velocity-independent pitch angle scattering. We're asked to find the parallel viscous stress. Recall the viscous stress tensor ${ }^{3}$

$$
\boldsymbol{\Pi}=\int m\left(\mathbf{w} \mathbf{w}-\frac{w^{2}}{3} \mathbf{I}\right) f_{1}(\mathbf{w}) d^{3} \mathbf{w}
$$

The parallel component is the gyro averaged part

$$
\mathbf{\Pi}_{\|}=\int m\left\langle\mathbf{w w}-\frac{w^{2}}{3} \mathbf{I}\right\rangle_{\theta}\left\langle f_{1}\right\rangle_{\theta} d^{3} \mathbf{w}
$$

You should recall ${ }^{4}$

$$
\begin{aligned}
\langle\mathbf{w} \mathbf{w}\rangle_{\theta}-\frac{w^{2}}{3} \mathbf{I} & =\left(w_{\|}^{2}-\frac{w_{\perp}^{2}}{2}\right)\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right) \\
& =\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right) w^{2} P_{2}(\xi)
\end{aligned}
$$

so we find

$$
\begin{equation*}
\boldsymbol{\Pi}_{\|}=2 \pi m\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right) \int w^{4} P_{2}(\xi)\left\langle f_{1}\right\rangle_{\theta} d \xi d w \tag{3}
\end{equation*}
$$

Now to calculate $f_{1}$ we consider the gyroaverage of Eq (2)

$$
C\left[\left\langle f_{1}\right\rangle\right]=f_{0}\left\{\left(\frac{w^{2}}{v_{T}^{2}}-\frac{5}{2}\right) w P_{1}(\hat{b} \cdot \nabla) \ln T+2 \frac{w^{2}}{v_{T}^{2}}\left[\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right) P_{2}+\frac{\mathbf{I}}{3}\right]: \nabla \mathbf{u}\right\}
$$

Since $\mathrm{Eq}(3)$ is a projection onto $P_{2}$, we can drop all the other components in $\left\langle f_{1}\right\rangle$ to find

$$
-3 \nu\left\langle f_{1}\right\rangle_{l=2}=2 f_{0} \frac{w^{2}}{v_{T}^{2}} P_{2}(\xi)\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right): \nabla \mathbf{u}
$$

This can be rearranged into

$$
\left\langle f_{1}\right\rangle_{l=2}=-\frac{2}{3 \nu}(\hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u}) f_{0} \frac{w^{2}}{v_{T}^{2}} P_{2}(\xi)
$$

${ }^{3}$ You may recall that the general pressure tensor is

$$
\mathbf{P}=\int m \mathbf{w} \mathbf{w} f(\mathbf{w}) d^{3} \mathbf{w}
$$

Viscosity is defined as the $\mathbf{P}=p \mathbf{I}+\boldsymbol{\Pi}$ where $p=\frac{1}{3} \operatorname{tr}(\mathbf{I})$. Note this is not just the off diagonal elements, but all terms which do not contribute to the trace. Since the 0th moment $f_{0}$ is an isotropic Maxwellian, it's viscous moment is 0 by construction. Therefore the viscous integral starts with $f_{1}$. This is why $\Pi / p \sim \epsilon$.
${ }^{4}$ Here are the missing steps

$$
\langle\mathbf{w} \mathbf{w}\rangle=\left(\begin{array}{ccc}
w_{\perp}^{2} / 2 & 0 & 0 \\
0 & w_{\perp}^{2} / 2 & 0 \\
0 & 0 & w_{\|}^{2}
\end{array}\right)=w_{\|}^{2} \hat{\mathbf{b}} \hat{\mathbf{b}}+\frac{w_{\perp}^{2}}{2}(\mathbf{I}-\hat{\mathbf{b}} \hat{\mathbf{b}})
$$

such that

$$
\langle\mathbf{w} \mathbf{w}\rangle-\frac{w^{2}}{3} \mathbf{I}=\left(\begin{array}{ccc}
w_{\perp}^{2} / 2 & 0 & 0 \\
0 & w_{\perp}^{2} / 2 & 0 \\
0 & 0 & w_{\|}^{2}
\end{array}\right)-\frac{w_{\perp}^{2}+w_{\|}^{2}}{3}\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)=\frac{1}{3}\left(\frac{w_{\perp}^{2}}{2}-w_{\|}^{2}\right)\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{array}\right)=\left(w_{\|}^{2}-\frac{w_{\perp}^{2}}{2}\right)\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right)
$$

and finally

$$
w_{\|}^{2}-\frac{w_{\perp}^{2}}{2}=w^{2}\left(\xi^{2}-\frac{1-\xi^{2}}{2}\right)=w^{2}\left(\frac{3 \xi^{2}-1}{2}\right)=w^{2} P_{2}(\xi)
$$

where we have used imcompressibility to write $\mathbf{I}: \nabla \mathbf{u}=\nabla \cdot \mathbf{u}=0$. Therefore

$$
\begin{aligned}
\boldsymbol{\Pi}_{\|} & =2 \pi m\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right)(\hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u}) f_{0} \frac{w^{2}}{v_{T}^{2}} P_{2}(\xi)\left[\int_{-1}^{1} P_{2}(\xi) P_{2}(\xi) d \xi\right] \int-\frac{2}{3 \nu} f_{0} \frac{w^{4}}{v_{T}^{2}} d w \\
& =-\frac{2 \pi}{3 \nu} m\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right)(\hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u})\left(\frac{2}{5}\right) 2 \int \frac{n}{\pi^{3 / 2} v_{T}^{5}} w^{6} e^{-\left(w / v_{T}\right)^{2}} d w \\
& =-\frac{4}{15 \sqrt{\pi}} \frac{n m v_{T}^{2}}{\nu}\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right)(\hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u})\left[\int_{0}^{\infty} x^{3} e^{-x} d x\right] \\
& =-\frac{8}{15 \sqrt{\pi}} \frac{\frac{1}{2} n m v_{T}^{2}}{\nu}\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right)(\hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u}) \Gamma\left(\frac{7}{2}\right) \\
& =-\frac{p}{\nu}\left(\hat{\mathbf{b}} \hat{\mathbf{b}}-\frac{\mathbf{I}}{3}\right) \hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u}
\end{aligned}
$$

## e) ion electron scaling

The electron parallel viscosity is a factor of $\sqrt{m_{e} / m_{i}}$ smaller

$$
\frac{\Pi_{\|}^{e}}{\Pi_{\|}^{i}} \sim \frac{m_{e} v_{T e}^{2}\left\langle f_{1}^{e}\right\rangle}{m_{i} v_{T i}^{2}\left\langle f_{1}^{i}\right\rangle} \sim \frac{\left\langle f_{1}^{e}\right\rangle}{\left\langle f_{1}^{i}\right\rangle} \sim \frac{\nu_{i i}}{\nu_{e i}} \sim \epsilon
$$

because $\left\langle f_{1}\right\rangle \propto 1 / \nu$, where $\nu$ is determined by $C[f]$.

## f) heat flow

The entropy equation is

$$
\begin{equation*}
\frac{3}{2} p \frac{d}{d t} \ln \left(\frac{p}{n^{\gamma}}\right)=-\boldsymbol{\Pi}: \nabla \mathbf{u}-\nabla \cdot \mathbf{q} \tag{4}
\end{equation*}
$$

We are told the plasma is incompressible, which implies that density is time independent

$$
\frac{d n}{d t}=-n \nabla \cdot \mathbf{v}=0
$$

This enables us to simplify Eq (4) into

$$
\frac{3}{2} n \frac{d T}{d t}=-\boldsymbol{\Pi}: \nabla \mathbf{u}-\nabla \cdot \mathbf{q}
$$

To leading order the RHS is dominated by parallel viscous stress. We can neglect heat flux because it is dominated by electron transport, which we just showed is at least $\sqrt{m_{e} / m_{i}}$ smaller. This leaves just

$$
\frac{3}{2} n \frac{d T}{d t}=-\boldsymbol{\Pi}_{\|}: \nabla \mathbf{u}
$$

Let us call that the temperature evolution equation. Substituting our expression for $\boldsymbol{\Pi}_{\|}$yields

$$
\frac{3}{2} n \frac{d T}{d t}=\frac{p}{\nu}(\hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u})^{2}
$$

where we have again used $\mathbf{I}: \nabla \mathbf{u}=\nabla \cdot \mathbf{u}=0$. To avoid viscous heating and to make temperature timeindependent, it is sufficient to let

$$
\hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u}=\nabla_{\|} u_{\|}=0
$$

The plasma can achieve such a steady state by orienting its $\mathbf{B}$ perpendicular to flow such that $u_{\|}=0$, or by allowing the parallel component of flow velocity to be constant along field lines. If neither of these can be achieved, the the temperature will increase via viscous heating

$$
\frac{d T}{d t}=\frac{2}{3}(\hat{\mathbf{b}} \hat{\mathbf{b}}: \nabla \mathbf{u})^{2} \frac{T}{\nu} \propto T^{5 / 2}
$$

since $\nu \propto 1 / T^{3 / 2}$. Even though we assumed $\nu$ to be constant in the collision operator, such a scaling still gives a valid instantaneous heating rate.


[^0]:    ${ }^{1}$ The 2D Laplacian in $(R, \phi)$ coordinates is given as part of the problem

    $$
    \nabla^{2}=\frac{1}{R}\left(R \frac{\partial}{\partial R}\right)+\frac{1}{R^{2}} \frac{\partial^{2}}{\partial \phi^{2}}
    $$

[^1]:    ${ }^{2}$ We should be careful here, because $\theta$ is not a coordinate in 1D. By taking the small angle approximation in a spherical space (const $v$ shell), we have effectively smeared the top of the sphere into a plane. That is how $\theta$ could play the role of $R$ in our 2D cylindrical Laplacian. As a result these integrals are actually 2D integrals

    $$
    \left\langle\theta^{2}\right\rangle=\frac{\int \theta^{2} f(\theta) d^{2} \theta}{\int f(\theta) d^{2} \theta}=\frac{\int \theta^{3} f(\theta) d \theta}{\int \theta f(\theta) d \theta}=\frac{\int x e^{-\alpha x} d x}{\int e^{-\alpha x} d x}=\frac{\Gamma(2) / \alpha^{2}}{\Gamma(1) / \alpha}=\frac{1}{\alpha}=2 \nu t
    $$

    The extra factor of 2 makes sense because random walks diffuse faster in higher dimensions. There is less probability of undoing a previous step.

