

We would like to solve the boundary value problem

$$\epsilon y'' + f(x)y' + f'(x)y = 0$$

where $0 < \epsilon \ll 1$. Our boundary conditions are

$$y(0) = 0$$

$$y(1) = 1$$

and $f(x)$ is defined to be both analytic and positive-definite on the interval $[0, 1]$.

The outer solution is straight forward. Recognize

$$(fy)' = 0$$

implies

$$y_{out} = \frac{f(1)}{f(x)}$$

where we have set the integration constant to satisfy $y(1) = 1$.

The inner solution requires a trick, since $fy' + y'f$ is a form where the Kruskal-Newton diagram (dominant balance) does not simplify anything, even in the boundary layer $x \rightarrow 0^+$. Instead let us recognize

$$\frac{d}{dx} [\epsilon y' + fy] = 0$$

This sets up a first order ODE with constant inhomogeneity

$$\epsilon y' + fy = B$$

We would like to set up a stretched parameter

$$x = \epsilon X$$

such that

$$\frac{dy}{dX} + fy = B$$

This is now solved by the general form

$$y(X) = \frac{1}{u(X)} \left[\int^X u(y) B dy + C \right]$$

where $u(x) = e^{\int^x f(t) dt}$ is related to the homogeneous solution (it's the inverse). Expanding the stretched coordinate to match our outer solution we see

$$\begin{aligned} y_{in}(x) &= e^{-\frac{1}{\epsilon} \int_0^x f(t) dt} \left[B \int_0^{x/\epsilon} u(y) dy + C \right] \\ &= e^{-\frac{1}{\epsilon} \int_0^x f(t) dt} \left[\frac{B}{\epsilon} \int_0^x \int_0^y f(t) dt dy + C \right] \end{aligned}$$

Since $f(x)$ is positive definite, the integral $\int_0^x f(t)dt$ is monotonically increasing.¹ For simplicity let us define a helper function

$$h(x) = \int_0^x \int_0^y f(t) dt dy$$

Therefore for $x \gg \epsilon$ the exponent vanishes, and $[\dots]$ is exponentially suppressed. On the other hand, that exponential is 1 for $x \rightarrow 0$. This enables us to use C to cancel the outer solution for matching $y(0) = 0$. We still have freedom to choose B , so let us set it to ϵ^2 to create scale separation in derivatives of the Bx/ϵ term. This yields

$$y_{uniform}(x) = e^{-\frac{1}{\epsilon} \int_0^x f(t)dt} \left[\epsilon h(x) - \frac{f(1)}{f(0)} \right] + \frac{f(1)}{f(x)}$$

¹Even if f is infinitesimal, we can adjust the factor out from to be $1/\epsilon^m$ for some sufficiently large exponent.