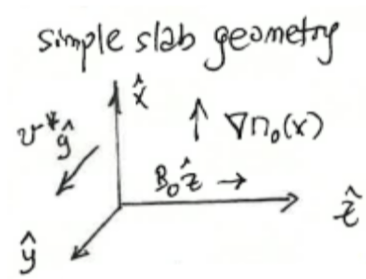
Here we look at a simplified derivation for electrostatic drift waves (oscillations having to do with drifts from inhomogeneous plasma ∇n_0. The picture you should have in your head is



where ∇n_0 increases in $+\hat{x}$, homogeneous B points in $+\hat{z}$, and the resulting drifts point along $\hat{y}$. We will also assume there is an electrostatic fluctuation [1]

$$\tilde{\phi} \sim e^{i(k_\perp y - \omega t)}$$

We suppose the resulting ion velocity perturbation is a combination of $E \times B$ and polarization drifts [2]

$$\tilde{v} = \frac{E}{B}\hat{x} - \frac{m}{qB^2}\frac{\partial E}{\partial t}\hat{y}$$

The continuity equation gives

$$\frac{\partial \tilde{n}}{\partial t} + n_0(\nabla \cdot \tilde{v}) + (\tilde{v} \cdot \nabla) n_0 = 0$$

Due to the background inhomogeneity, both space derivatives make a first order contribution

$$(\tilde{v} \cdot \nabla) n_0 = \frac{E}{B}\frac{dn_0}{dx} - \frac{m\dot{E}}{qB^2}\frac{dn_0}{dy} = -ik_\perp \left(\frac{e\phi}{m_i \Omega_i}\right)\frac{dn_0}{dx}$$

$$\nabla \cdot \tilde{v} = \partial_x \left(\frac{E}{B}\right) - \partial_y \left(\frac{m\dot{E}}{qB^2}\right) = -ik_\perp^2 \left(\frac{e\phi}{m_i \Omega_i}\right)\frac{\omega}{\Omega_i}$$

[1] In the more general case, we add a parallel component to the oscillation

$$\tilde{\phi} = \phi(x) e^{i(k_\perp y + k_\parallel z - \omega t)}$$

as well as an x-dependent amplitude. The resulting dispersion relation from doing a fluid momentum calculation is

$$1 - \frac{C_s^2 k_\parallel^2}{2\omega^2} - \frac{\omega_e^*}{\omega} + \frac{1}{2}(k_\perp \rho_i)^2 = 0$$

You may compare this with the answer we derive in Eq (3).

[2] Here's my favorite derivation of polarization drift

$$v_1 = \frac{E \times B}{B^2}$$

$$F_1 = m\dot{v}_1$$

$$v_2 = \frac{F_1 \times B}{qB^2} = -\frac{m}{qB^2}\hat{b} \times (\dot{E} \times \hat{b})$$

We see that the polarization drift results from E oscillating in a direction perpendicular B. Although this is derived recursively from the $E \times B$ drift, it is neither asymptotically larger or smaller than $E \times B$. Instead it is proportional to $\propto \omega$ the oscillation frequency.

Then the continuity equation says

$$\left(\frac{\tilde{n}}{n_0}\right)_i + k_\perp^2 \left(\frac{e\phi}{m_i\Omega_i}\right)\frac{1}{\Omega_i} + k_\perp \left(\frac{e\phi}{m_i\Omega_i}\right)\frac{1}{\omega}\frac{d}{dx}\ln n_0 = 0 \tag{1}$$

For electrons, we assume they are adiabatic. Therefore the density profile satisfies a Boltzmann distribution

$$n(x) = n_0 e^{e\tilde{\phi}/T_e}$$

Taylor expanding both sides shows

$$n_0 + \tilde{n} = n_0\left(1 + \frac{e\tilde{\phi}}{T_e}\right)$$

which yields

$$\left(\frac{\tilde{n}}{n_0}\right)_e = \frac{e\tilde{\phi}}{T_e} \tag{2}$$

Now we assert quasineutrality

$$0 = \sum_s q_s n_s = e\left[(n_0+\tilde{n})_i - (n_0+\tilde{n})_e\right] = -en_0\left[\left(\frac{\tilde{n}}{n_0}\right)_e - \left(\frac{\tilde{n}}{n_0}\right)_i\right]$$

Plugging in Eq (1) and Eq (2) shows

$$\frac{e\tilde{\phi}}{T_e}\left\{1 + k_\perp^2\left(\frac{T_e}{m_i\Omega_i^2}\right) + \frac{k_\perp}{\omega}\left(\frac{T_e}{m_i\Omega_i}\right)\frac{d}{dx}\ln n_0\right\} = 0$$

Setting the quanitity in $\{\dots\} = 0$ yields the dispersion relation. We can further polish the result by recognizing

$$k_\perp^2\left(\frac{T_e}{m_i\Omega_i^2}\right) = \frac{k_\perp^2 C_s^2}{2\Omega_i^2} = \frac{1}{2}\left(k_\perp\rho_i\right)^2$$

and

$$\frac{T_e}{m_i\Omega_i} = -\frac{T_e}{m_e\Omega_e} = -\frac{v_{Te}^2}{2\Omega_e}$$

while recalling the electron drift wave frequency [3]

$$\omega_e^* = k_\perp\left(\frac{v_{Te}^2}{2\Omega_e}\right)\frac{d}{dx}\ln n_0$$

Now the updated drift wave dispersion relation is

$$1 - \frac{\omega_e^*}{\omega} + \frac{1}{2}\left(k_\perp\rho_i\right)^2 = 0 \tag{3}$$

[3] to derive the drift frequency recall

$$J \times B = \nabla p$$
$$J = qnv$$
$$\nabla p = T\nabla n$$

then

$$v_s^* = \frac{T}{qB}\nabla \ln n = \frac{v_{Ts}^2}{2\Omega_s}\nabla\ln n$$

it remains only to define

$$\omega_s^* = k\cdot v_s^*$$