1 Mirrors with ξ-dependent Source

a) classical mirror

This starts out as your typical mirror problem. The key constraint is the ratio of parallel to perpendicular kinetic energy at the mid plane. We apply conservation of energy to see that

$$W_0^{\parallel} + W_0^{\perp} < W_f^{\perp}$$

Tives the trapping condition. But the first adiabatic moment $\mu = W^{\perp}/B$ is conserved. Thus the mirror ratio enters our constraint

$$\frac{W_0^{\parallel}}{W_0^{\perp}} + 1 < \frac{B_f}{B_0} = R_m$$

Now the kinetic energies can be expressed as a function of pitch angle $\xi = v_{\parallel}/v$. We see

$$\frac{W_0^{\parallel}}{W_0^{\perp}} = \frac{\xi^2}{1-\xi^2} < R_m - 1$$

So isolating ξ we find that the trapping condition in pitch-angle space is

$$\xi < \xi_c = \sqrt{1 - \frac{1}{R_m}}$$

We see that in the limit $R_m \to 1$ nothing is trapped, and $R_m \to \infty$ means everything is trapped as expected.

b) pitch-angle scattering

Now let us introduce a collision operator

$$C[f] = \frac{\nu}{2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial f}{\partial \xi}\right] = \nu L[f]$$

This Lorentz operator isotropizes the pitch angle, but does not alter energy. Particles will scatter into the loss cone ($\xi > \xi_c$). To compensate we introduce a steady-state source

$$S(v,\xi) = \begin{cases} \frac{3\dot{n}}{8\pi v_0^2 \xi_c^3}\delta(v-v_0)(\xi_c^2 - \xi^2) & \text{for } |\xi| < \xi_c \\ 0 & \text{for } |\xi| \geq \xi_c \end{cases}$$

This is a monoenergetic source $\delta(v-v_0)$ which preferentially introduces particles with large pitch angle (small $v_{\parallel} = v\xi$). To solve for the equilibrium, we write down a steady-state kinetic equation

$$0 = \frac{df}{dt} = C[f] + S$$

Since P_0 is involved, the Legendre basis will not be of much use, so we go for dirrect integration.

$$\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial f}{\partial \xi}\right] = -\frac{2}{\nu}A(\xi_c^2 - \xi^2)$$

The boundary conditions are $f(\pm\xi_c) = 0$ and $f'(0) = 0$. Thus

$$(1-\xi^2)f'(\xi) = -\frac{2}{\nu}A\xi\left(\xi_c^2 - \frac{\xi^2}{3}\right)$$

which lets us set up

$$0 - f(\xi) = -\frac{A}{\nu}\int_\xi^{\xi_c} \frac{2\xi}{1-\xi^2}\left(\xi_c^2 - \frac{\xi^2}{3}\right) d\xi$$

To compute this integral we play the following trick: $\xi^2 = 1 - (1-\xi^2)$. This enables us to write

$$f(\xi) = \frac{A}{\nu}\left[\left(\xi_c^2 - \frac{1}{3}\right)\int_\xi^{\xi_c} \frac{2\xi}{1-\xi^2} d\xi + \int_\xi^{\xi_c} 2\xi\, d\xi\right]$$

We can also cast the constant coefficient as a Legendre Polynomial

$$P_2(\xi_c) = \frac{3\xi_c^2 - 1}{2}$$

Together this yields

$$f(\xi) = \frac{2A}{3\nu}\left[P_2(\xi_c)\ln\left(\frac{1-\xi^2}{1-\xi_c^2}\right) + \frac{\xi_c^2 - \xi^2}{2}\right]$$

Adding back the constant we have

$$f_{eq}(v,\xi) = \frac{\dot{n}}{4\pi\nu v_0^2 \xi_c^3}\delta(v-v_0)\left[P_2(\xi_c)\ln\left(\frac{1-\xi^2}{1-\xi_c^2}\right) + \frac{\xi_c^2 - \xi^2}{2}\right]$$

for $|\xi| < \xi_c$ and $f_{eq} = 0$ otherwise.

c) pressure anistropy

Now we would like to copmute the pressure anisotropy as a fucntion of R_m. Let us start by recognizing that $\Delta p = p_\perp - p_\|$ depends on the second Legendre polynomial

$$\begin{aligned}
\Delta p &= \int m\left(\frac{w_\perp^2}{2} - w_\|^2\right) f_{eq}\, d^3w \\
&= 2\pi m \int w^2 \left(\frac{w_\perp^2}{2} - w_\|^2\right) f_{eq}\, d\xi\, dw \\
&= -2\pi m \int w^4 P_2(\xi) f_{eq}\, d\xi\, dw
\end{aligned}$$

Next we can substitute and integrate over the δ-function

$$\Delta p = -\frac{\dot{n}v_0^2}{2\nu\xi_c^3}P_2(\xi_c)\int_{-\xi_c}^{\xi_c} P_2(\xi)\left[\ln\left(\frac{1-\xi^2}{1-\xi_c^2}\right) + \frac{\xi_c^2 - \xi^2}{2}\right] d\xi$$

These integrals are given to us in the problem statement

$$\int_{-\xi_c}^{\xi_c} P_2(\xi)\ln\left(\frac{1-\xi^2}{1-\xi_c^2}\right) d\xi = -\frac{2}{3}\xi_c^3$$

$$\int_{-\xi_c}^{\xi_c} P_2(\xi)\left(\frac{\xi_c^2-\xi^2}{2}\right) d\xi = -\frac{1}{3}\int_{-\xi_c}^{\xi_c} P_2(\xi)\left[P_2(\xi)-P_2(\xi_c)\right] d\xi$$
$$= \frac{4}{15}\left[\frac{P_2(\xi_c)}{2}-1\right]\xi_c^3$$

Now combining these results we see

$$-\frac{2}{3}P_2(\xi_c)+\frac{4}{15}\left[\frac{P_2(\xi_c)}{2}-1\right] = -\frac{4}{5}\xi_c^2$$

Therefore

$$\Delta p = \frac{2}{5}\dot{n}\frac{v_0^2}{\nu}\xi_c^2 = \frac{2}{5}\dot{n}\frac{v_0^2}{\nu}\left(1-\frac{1}{R_m}\right)$$

When $R_m \to 1$ and there is a weak mirror, $\Delta p \to 0$, but when $R_m \to \infty$ the pressure anisotropy is asymptotically constant. Since particles with fast parallel velocity are preferentially lost, it makes sense that $\Delta p = p_\perp - p_\parallel > 0$.

2 Braginskii Anistropy

d)

We consider the Braginskii ion pressure anistropy. Let us start with the ion Maxweillian

$$f_0(t,\mathbf{r},w) = \frac{n(t,\mathbf{r})}{\pi^{3/2}v_T^3}e^{-(w/v_T)^2}$$

where $v_T^2 = 2T(t,\mathbf{r})/m$. Now the kinetic equation which governs the first order correction is

$$\Omega\frac{\partial f_1}{\partial\theta} + C[f_1] = f_0\left[\left(\frac{w^2}{v_T^2}-\frac{5}{2}\right)\mathbf{w}\cdot\nabla\ln T + 2\left(\frac{\mathbf{ww}}{v_T^2}-\frac{\mathbf{I}}{3}\frac{w^2}{v_T^2}\right):\nabla\mathbf{u}\right] \tag{1}$$

We are interested in solving for the Braginskii ion pressure anisotropy

$$\Delta p = p_\perp - p_\parallel$$

The partial pressures can be defined

$$p_\parallel = \int m w_\parallel^2 f_1\, d^3\mathbf{w}$$
$$p_\perp = \int m\frac{w_\perp^2}{2} f_1\, d^3\mathbf{w}$$

which gives us a short-cut since the gyro average is defined

$$\langle f_1\rangle_\theta = \frac{1}{2\pi}\int f_1\, d\theta$$

So we only need to compute

$$p_\parallel = 2\pi m\int_0^\infty\int_{-1}^1 w^4\xi^2\langle f_1\rangle_\theta\, d\xi\, dw$$
$$p_\perp = \pi m\int_0^\infty\int_{-1}^1 w^4(1-\xi^2)\langle f_1\rangle_\theta\, d\xi\, dw$$

The second short-cut is to take the difference

$$\Delta p = -2\pi m \int_0^\infty \int_{-1}^1 w^4 \left(\frac{3\xi^2 - 1}{2}\right) \langle f_1 \rangle_\theta \, d\xi \, dw \tag{2}$$

and recognize the Legendre polynomial

$$P_2(\xi) = \frac{3\xi^2 - 1}{2}$$

Now let's cash in on that gyroaverage. Eq (1) becomes

$$C[\langle f_1 \rangle_\theta] = f_0 \left[\left(\frac{w^2}{v_T^2} - \frac{5}{2} \right) \langle \mathbf{w} \rangle_\theta \cdot \nabla \ln T + 2 \left(\frac{\langle \mathbf{ww} \rangle_\theta}{v_T^2} - \frac{\mathbf{I}}{3} \frac{w^2}{v_T^2} \right) : \nabla \mathbf{u} \right]$$

We should readily recognize

$$\langle \mathbf{w} \rangle_\theta = w_\parallel \hat{b}$$

$$\langle \mathbf{ww} \rangle_\theta = w_\parallel^2 \hat{b}\hat{b} + \frac{w_\perp^2}{2} \left(\mathbf{I} - \hat{b}\hat{b} \right)$$

and compute

$$\left(w_\parallel^2 \hat{b}\hat{b} + \frac{w_\perp^2}{2} \right) - \left(\frac{w_\parallel^2 + w_\perp^2}{3} \right) \mathbf{I} = \left(w_\parallel^2 - \frac{w_\perp^2}{2} \right) \left(\hat{b}\hat{b} - \frac{\mathbf{I}}{3} \right) = w^2 P_2(\xi) \left(\hat{b}\hat{b} - \frac{\mathbf{I}}{3} \right)$$

Now the kinetic equation becomes

$$C[\langle f_1 \rangle_\theta] = f_0 \left[\left(\frac{w^2}{v_T^2} - \frac{5}{2} \right) w P_1(\xi) \nabla_\parallel \ln T + 2 P_2(\xi) \frac{w^2}{v_T^2} \left(\hat{b}\hat{b} - \frac{\mathbf{I}}{3} \right) : \nabla \mathbf{u} \right]$$

This is very nice because, using the Lorentz operator for ion collisions $C[f] = \nu L[f]$, we can expand

$$\langle f_1 \rangle_\theta = \sum_{l=0}^{\infty} u_l(v) P_l(\xi)$$

since the Legendre polynomials $P_l(\xi)$ form the eigenbasis for the Lorentz operator with eigenvalues

$$L[P_l] = -\frac{l(l+1)}{2} P_l$$

Therefore

$$C[\langle f_1 \rangle_\theta] = \nu L \left[\langle f_1 \rangle_\theta \right] = -\nu \left(u_1 P_1 + 3 u_2 P_2 \right)$$

to match the RHS. Then equating the orthogonal polynomials shows

$$u_1 = -\frac{f_0}{\nu} \left(\frac{w^2}{v_T^2} - \frac{5}{2} \right) w \nabla_\parallel \ln T$$

$$u_2 = -\frac{2 f_0}{3\nu} \frac{w^2}{v_T^2} \left(\hat{b}\hat{b} - \frac{\mathbf{I}}{3} \right) : \nabla \mathbf{u}$$

Now lets return to our pressure anistropy Eq (2)

$$\Delta p = -2\pi m \int_0^\infty \int_{-1}^1 w^4 P_2(\xi) \langle f_1 \rangle_\theta \, d\xi \, dw$$

The Legendre polynomials form an orthonormal basis

$$\int_{-1}^{1} P_l(\xi)P_m(\xi)d\xi = \frac{\delta_{lm}}{l+1/2}$$

So we find

$$\Delta p = -\frac{4}{3}\cdot\frac{2}{5}\pi m\int_0^\infty w^4 u_2\,dw$$

Now let us substitute the Maxwellian for f_0 in $u_2(w)$ and find

$$\begin{aligned}
\Delta p &= -\frac{8}{15}\pi m\int_0^\infty w^4\left(-\frac{2}{3\nu}\right)\frac{w^2}{v_T^2}\left(\hat{b}\hat{b}-\frac{\mathbf{I}}{3}\right):\nabla\mathbf{u}\left[\frac{n_0}{\pi^{3/2}v_T^3}e^{-(w/v_T)^2}\right]dw \\
&= \frac{8}{15\sqrt{\pi}}\left(\frac{2mn_0v_T^2}{3\nu}\right)\left(\hat{b}\hat{b}-\frac{\mathbf{I}}{3}\right):\nabla\mathbf{u}\int_0^\infty x^6e^{-x^2}dx \\
&= \frac{8}{15\sqrt{\pi}}\left(\frac{2n_0T}{\nu}\right)\left(\hat{b}\hat{b}-\frac{\mathbf{I}}{3}\right):\nabla\mathbf{u}\int_0^\infty y^{5/2}e^{-y}dy \\
&= \frac{8}{15\sqrt{\pi}}\left(\frac{2p}{\nu}\right)\left(\hat{b}\hat{b}-\frac{\mathbf{I}}{3}\right):\nabla\mathbf{u}\left[\Gamma\left(\frac{7}{2}\right)\right] \\
&= \frac{p}{\nu}\left(\hat{b}\hat{b}-\frac{\mathbf{I}}{3}\right):\nabla\mathbf{u}
\end{aligned}$$

where we recall $\Gamma(7/2) = 15\sqrt{\pi}/8$. [1] This is the Braginskii ion pressure anisotropy.

e)

We are told that Mikhailovskii-Tsypin order a weakly collisional, magnetized plasma (as opposed to Braginskii which is strongly collisional) and get an extra term

$$\Delta p = \left\{\frac{p}{\nu}\left(\hat{b}\hat{b}-\frac{\mathbf{I}}{3}\right):\nabla\mathbf{u}\right\} - \frac{1}{\nu}\left\{q_\perp(\nabla\cdot\hat{b}) + \frac{1}{3}\nabla\cdot\left[\hat{b}(q_\perp - q_\parallel)\right]\right\}$$

We don't see these extra terms because heat flux is next order in Braginskii transport. Let's call the first term Δp_1. Its ordering goes like

$$\Delta p_1 \sim \frac{p}{\nu_{ii}}\frac{u_i}{L} \sim p\frac{u_i}{v_{Ti}}\frac{\lambda_{ii}}{L}$$

Let's call the second term Δp_2. Noting that $q \sim pv_T(\lambda_{ii}/L)$ we find

$$\Delta p_2 \sim \frac{q}{\nu_{ii}L} \sim p\frac{v_{Ti}}{v_{Ti}}\frac{\lambda_{ii}}{L}\frac{\lambda_{ii}}{L}$$

The key difference between Branginskii and Mihailovskii is high-flow vs low-flow ordering

$$\left(\frac{u_i}{v_{Ti}}\right)_B \sim 1 \qquad\qquad \left(\frac{u_i}{v_{Ti}}\right)_M \sim \epsilon$$

Thus we see Δp_2 *always* scales like $\sim p\epsilon^2$, while Δp_1 scales like $\sim p\epsilon^2$ only for Mikhailovskii, but $\sim p\epsilon$ for Bragniskii. Since the second term is next order for Braginskii, only the first term appears.

[1] There's a quick mnemonic for this. All you need is $\Gamma(1/2) = \sqrt{\pi}$ and the factorial property $\Gamma(n+1) = n\Gamma(n)$. Then

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$