This problem looks at magnetized warm plasma waves. We are given the dielectric tensor

$$\epsilon = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

The electrostatic dispersion relation is [1]

$$D_\phi = \langle k|\epsilon|k\rangle = 0$$

Considering $\mathbf{k} = (k_\perp, 0, k_\parallel)$ this shows

$$k_\perp^2 S + k_\parallel^2 P = 0 \tag{1}$$

Now we would like to find the dispersion relation $\omega(\mathbf{k})$ assuming

- $\omega \approx \Omega_i$ (near ion cyclotron resonance)
- $C_s \ll \omega/k_\parallel \ll v_{Te}$ (warm electrons, cold ions)
- $k\lambda_{De} \ll 1$ (wavelength is large compared to sheath)

We should also note that the problem states Ilya's convention $v_{Te}^2 = T_e/m_e$, which implies $C_s^2 = T_e/m_i$. [2] Also, note $T_e = T_e^\parallel$. Since we are near ion resonance, $L \gg R$. Therefore

$$S = \frac{L+R}{2} \approx \frac{L}{2} = \frac{1}{2}\left(1 - \frac{\omega_{pi}^2}{\omega(\omega - \Omega_i)} - \frac{\omega_{pe}^2}{\omega(\Omega + |\Omega_e|^2)}\right)$$
$$= -\frac{\omega_{pi}^2}{2\Omega_i(\omega - \Omega_i)}$$

The parallel propagation corresponds to Langmuir waves. Compared to the cold magnetized fluid case, the warm magentized fluid has an extra $\gamma(k_\parallel v_{Te})^2$ [3]

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \gamma k_\parallel^2 v_{Ts}^2}$$
$$= 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\gamma k_\parallel^2 v_{Te}^2}$$
$$\approx \frac{\omega_{pe}^2}{\gamma k_\parallel^2 v_{Te}^2}$$

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[1] Here's a quick derivation, start from Maxwell's equation in matter:

$$\nabla \cdot D = 4\pi\rho$$

then expand $D = \epsilon E$ with $E = -\nabla\phi$

$$-\nabla \cdot (\epsilon \nabla \phi) = 4\pi\rho_0$$

For waves we interested in the homogeneous solution. So linearizing gives

$$D_\phi \phi = k \cdot (\epsilon k)\phi = 0$$

This can be written as an inner product

$$D_\phi = \langle k|\epsilon|k\rangle$$

[2] not to be confused with Matt's convention $v_{Te} = 2T_e/m_e$ and $C_s^2 = 2T_e/m_i$. Fortunately, if you stay consistent, either will give the same answer.

[3] Let us give the proof as an appendix.

We can set $\gamma = 1$ and $v_T = \omega_p \lambda_D$ to write

$$P = \left(\frac{1}{k_\parallel \lambda_{De}}\right)^2$$

Next we substitute both into the dispserion relation Eq (1). This shows

$$0 = -\frac{\omega_{pi}^2 k_\perp^2}{2\Omega_i(\omega - \Omega_i)} + \frac{1}{\lambda_{De}^2}$$

Note that the $k_\parallel$ dependence has cancelled. Rearranging terms shows

$$\begin{aligned}\omega &= \Omega_i + \frac{(\lambda_{De}\omega_{pi}k_\perp)^2}{2\Omega_i} \\ &= \Omega_i + \frac{C_s^2 k_\perp^2}{2\Omega_i}\end{aligned}$$

where we have used $(\lambda_{De}\omega_{pi})^2 = \frac{T_e}{m_i} = C_s^2$. Now squaring the relation shows

$$w^2 = \Omega_i^2 + C_s^2 k_\perp^2 + \frac{1}{4}\left(\frac{C_s}{\Omega_i/k_\perp}\right)^2$$

We assumed $C_s \ll \omega/k_\parallel$. We have $\omega \approx \Omega_i$. So supposing $k_\perp$ is only slightly larger than $k_\parallel$, we can drop the last term. Therefore

$$w^2 \approx \Omega_i^2 + C_s^2 k_\perp^2$$

Appendix: Warm Dielectric Tensor

A short derivation involves the usual suspects

1. momentum equation
$$\frac{\partial \mathbf{v}}{\partial t} = \frac{q}{m}\mathbf{E} - \frac{\nabla p}{mn}$$
2. continuity equation
$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$
3. isoentropic closure
$$\frac{d}{dt}\left[\frac{p}{n^\gamma}\right] = 0$$

We rewrite the last condition as

$$\frac{1}{n^\gamma}\left(\frac{\partial p}{\partial t} - \gamma T \frac{\partial n}{\partial t}\right) = 0$$

and take a derivative on the momentum equation to find

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{v}) = \frac{q}{m}\mathbf{E}$$

This can be lienarized to see

$$\left(-\omega^2 + \gamma v_T^2 \mathbf{k}\mathbf{k}\right)\mathbf{v} = -i\omega\frac{q}{m}\mathbf{E}$$

Now substituting $\mathbf{J} = qn\mathbf{v}$ we find

$$\mathbf{J} = \frac{i\omega}{4\pi}\left(\frac{\omega_p^2}{\omega^2 - \gamma v_T^2 k^2}\right)\mathbf{E}$$

Now $\mathbf{J} = \sigma\mathbf{E}$ and $\chi = \frac{4\pi i}{\omega}\sigma$ so

$$\chi_{\|} = -\frac{\omega_p^2}{\omega^2 - \gamma v_T^2 k^2}$$

and

$$\epsilon = 1 + \sum_s \chi_s$$

gives

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \gamma_s v_{Ts}^2 k^2}$$