Magnetic Mirror with Gravity (2021 - I.1) **Tony M. Qian : 3 May 2021**

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a)

Let us start by considering a generic mirror. The trapping condition can be found by conservation of energy

$$W_0^{\|} + W_0^{\perp} < W_f^{\perp} + 0$$

From here we normalize by the midplane perpendicular energy to find

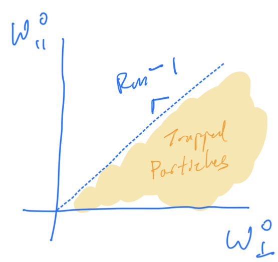
$$\frac{W_0^{\|}}{W_0^{\perp}} + 1 < \frac{W_f^{\perp}}{W_0^{\perp}} = \frac{B_f}{B_0} = R_m$$

Therefore the trapping condition is

$$\frac{W_0^{\|}}{W_0^{\perp}} < R_m - 1 \tag{1}$$

b)

We can show this pictorially as



Here we see the usual cast of characters. The ratio of energy at the midplane determines the trapping conditions everywhere. Particles with arbitrarily large $W_0^{\|}$ will always escape. Larger mirror ratio R_m indicates a larger region of trapped particles. Conversely all particles escape in the limit $R_m \to 1$.

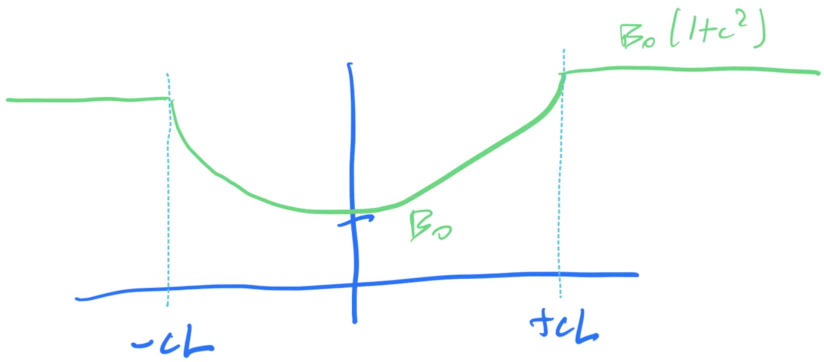
c)

Now we specify a particular field profile

$$\mathbf{B} = \begin{cases} B_0\left(1 + \frac{z^2}{L^2}\right)\hat{z} & (z/L)^2 < c^2 \\ B_0(1 + c^2)\hat{z} & (z/L)^2 > c^2 \end{cases}$$

This means the mirror has total length $\Delta z = 2cL$ and that the mirror ratio is $R_m = 1 + c^2$.
The turning points can be found by applying conservation of energy again

$$W_0^{\|} + W_0^{\perp} = W_z^{\perp}$$



Doing the same normalization

$$\frac{W_0^{\parallel}}{W_0^{\perp}} + 1 = \frac{W_z^{\perp}}{W_0^{\perp}} = 1 + \left(\frac{z_T}{L}\right)^2$$

Therefore

$$\left(\frac{z_T}{L}\right)^2 = \frac{W_0^{\parallel}}{W_0^{\perp}} \tag{2}$$

Note that the maximum turning points are $z_T = \pm cL$, which is larger than L. For some reason the field is set up such that L is not the length of the mirror, but instead the length after which the field strength increases a quadratic multiple.

d)

Now we impose an axially gravitational field

$$\mathbf{g} = -g\hat{z}$$

This makes it easier to escape the bottom, and harder to escape the top. So to write a trapping condition, we can focus on the bottom. Let us start once more from energy conservation

$$W_0^{\parallel} + W_0^{\perp} = W_z^{\perp} + mgz$$

Since it is more difficult to confine the bottom, we will use that as the trapping condition

$$\frac{W_0^{\parallel}}{W_0^{\perp}} + 1 < R_m - \frac{mgcL}{W_0^{\perp}}$$

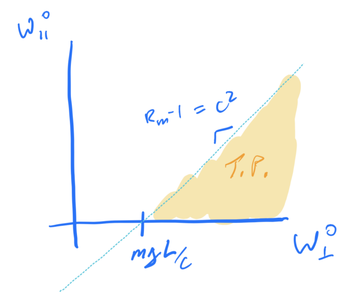
If the ratio of parallel to perpendicular energies exceeds this threshold at the midplane, then the particle will escale through $z < -cL$. But we also know from part (c) that $R_m = 1 + c^2$. Thus the trapping condition is

$$\begin{aligned}\frac{W_0^{\parallel}}{W_0^{\perp}} &< (R_m - 1) - \frac{mgcL}{W_0^{\perp}} \\ &= c^2\left(1 - \frac{mgL}{cW_0^{\perp}}\right)\end{aligned}$$

Notice that the first line reduces to our original trapping condition in Eq (1) in the limit $g \to 0$. The second line shows that there is now a minimum perpendicular energy requirement for trapping

$$W_0^{\perp} > \frac{1}{c}(mgL) \tag{3}$$

This can be interpretted as the minimum perpendicular energy needed to offset the parallel accleration picked up from gravity. This is in addition to parallel to perpendicular midplane energy ratio constraint, which has the same slope as before $R_m - 1 = c^2$.



e)

Now we wish to find the turning points. This will be a function of the midplane energies, as different particles will execute different orbits. This time we find

$$\frac{W_0^\parallel}{W_0^\perp} = \left(\frac{z_T}{L}\right)^2 + \frac{mgz_T}{W_0^\perp}$$

This is a quardatic equation

$$\left(\frac{z_T}{L}\right)^2 + 2\left(\frac{mgL}{2W_0^\perp}\right)\frac{z_T}{L} - \frac{W_0^\parallel}{W_0^\perp} = 0$$

So we find

$$\frac{z_T}{L} = -\frac{mgL}{2W_0^\perp} \pm \sqrt{\left(\frac{mgL}{2W_0^\perp}\right)^2 + \frac{W_0^\parallel}{W_0^\perp}}$$

First note that this recovers the original trapping condition Eq (2) in the limit $g \to 0$. Relative to that case, the top and bottom turning points are unformly shifted by $-z_g/2$ where

$$z_g = \frac{mgL^2}{W_0^\perp}$$

This lets us recast the new turnning points as

$$z_H = -\frac{z_g}{2} + \sqrt{z_T^2 + \left(\frac{z_g}{2}\right)^2} \tag{4}$$

$$z_L = -\frac{z_g}{2} - \sqrt{z_T^2 + \left(\frac{z_g}{2}\right)^2} \tag{5}$$

From here we can see a relation

$$z_g = -(z_H + z_L) > 0 \tag{6}$$

The signs just indicate that $|z_L| > z_H$, which we expect because the mirror shifted net down in the direction of gravity. We also see

$$\left(\frac{z_H - z_L}{2}\right)^2 = z_T^2 + \left(\frac{z_g}{2}\right)^2$$

Substituting Eq (6) we see

$$z_T^2 = -z_H z_L > 0 \tag{7}$$

f)

Next we suppose that gravity is adiabatically increased from $g_i = 0$ to $g_f = g$.

Since $\mu = W_\perp / B$ is an adiabtic invaraint, the fact that B does not change while ramping gravity implies that $W_\perp^0$ also stays constant.

g)

To analyze the effect on $W_0^\parallel$ we should introduce the second adiabatic invariant

$$J = \oint m v_\parallel dl$$

where the integral is taken over the cyclic bounce path. [1] Let us rewrite this in a dimensionless form in hopes that it will simplify algebra downstream

$$K = \frac{1}{L} \int_{z_L}^{z_H} \sqrt{\frac{W_z^\parallel}{W_0^\perp}}\, dz \tag{8}$$

Here $z_H > 0$ and $z_L < 0$ are the turning points from (e), $W_0^\perp$ is a constant thanks to (f), and $W_z^\parallel = W^\parallel(z)$ is a function to be determined. We can constrain this function using energy conservation

$$W^\parallel(z) + W^\perp(z) + mgz = W_0^\parallel + W_0^\perp$$

Normalizing by $W_0^\perp$ we see

$$\frac{W_z^\parallel}{W_0^\perp} + \left(1 + \frac{z^2}{L^2}\right) + \frac{mgz}{W_0^\perp} = \left(\frac{z_T}{L}\right)^2 + 1$$

Therefore

$$\begin{aligned}
\frac{W_z^\parallel}{W_0^\perp} &= \left(\frac{z_T}{L}\right)^2 - \frac{mgz}{W_0^\perp} - \frac{z^2}{L^2} \\
&= \frac{1}{L^2}\left(z_T^2 - z_g z - z^2\right) \\
&= \frac{1}{L^2}\left[(-z_H z_L) + (z_H + z_L)z - z^2\right] \\
&= \frac{1}{L^2}(z_H - z)(z - z_L)
\end{aligned}$$

where we used Eq (6) and Eq (7) from (e). Each quantity in paraentheses is positive, since z bounces between $z_H > 0$ and $z_L < 0$. Therefore the product is positive, as it should be. Now our adiabatic integral becomes

$$K = \frac{1}{L^2} \int_{z_L}^{z_H} \sqrt{(z_H - z)(z - z_L)}\, dz = \frac{\pi}{8}\left(\frac{z_H - z_L}{L}\right)^2$$

[1] In general a conserved ‘action’ can be defined for any coordinate undergoing quasi periodic motion

$$J = \oint p dq$$

This is the basis of Hamilton and Jacobi’s action-angle mechanics.

This is an elliptic integral, which is given in the problem statement. K is an adiatibatic invariant, but z_H and z_L each change as functions of g. Substituting the difference between Eq (4) and Eq (5) we find

$$K = \frac{\pi}{2}\left[\left(\frac{z_T}{L}\right)^2 + \left(\frac{z_g/2}{L}\right)^2\right]$$

Therefore

$$K_i = \frac{\pi}{2}\left[\frac{W_{0,i}^{\parallel}}{W_0^{\perp}} + \left(\frac{mg_i L}{2W_0^{\perp}}\right)^2\right] \tag{9}$$

$$K_f = \frac{\pi}{2}\left[\frac{W_{0,f}^{\parallel}}{W_0^{\perp}} + \left(\frac{mg_f L}{2W_0^{\perp}}\right)^2\right] \tag{10}$$

Letting

$$K_i = K_f$$

and recognizing $g_i = 0$ and $g_f = g$ we find

$$W_{0,i}^{\parallel} - W_{0,f}^{\parallel} = \frac{(mgL)^2}{4W_0^{\perp}} = \frac{1}{4}mgz_g$$

This means the net change is negative

$$\Delta W_0^{\parallel} = \left(W_{0,f}^{\parallel} - W_{0,i}^{\parallel}\right) = -\frac{1}{4}mgz_g < 0 \tag{11}$$

As gravity is slowly turned on, the midplane parallel energy decreases. This can be understood from the fact that $|z_L| > z_H$ means $(z = 0)$, the midplane of the magnetic field, is no longer the midplane of the bounce trajectories. Since the lower path is longer, particles have to fight gravity to get to the magnetic midplane. Thus the parallel energy measured at that location is decreased. [2]

i)

Next we are interested in particles that are initially trapped and later detrapped by the adiabatically activated g field.

For $g_i = 0$ the picture looks like (b), for $g_f = g$ the picture is described by (d-g). We can think of two changes in parallel energy. One is the change in the trapping condition

$$-\Delta W_1^{\parallel} = (R_m - 1)\Delta W^{\perp} = c^2\left(\frac{mgL}{c}\right) = mgLc$$

This is the new, more strict trapping condition we considered in Eq (3) from part (d). However, there is a change in particle energy that competes to stay up to date with the new rules

$$-\Delta W_2^{\parallel} = \frac{1}{4}mgz_g = \frac{(mgL)^2}{4W_0^{\perp}}$$

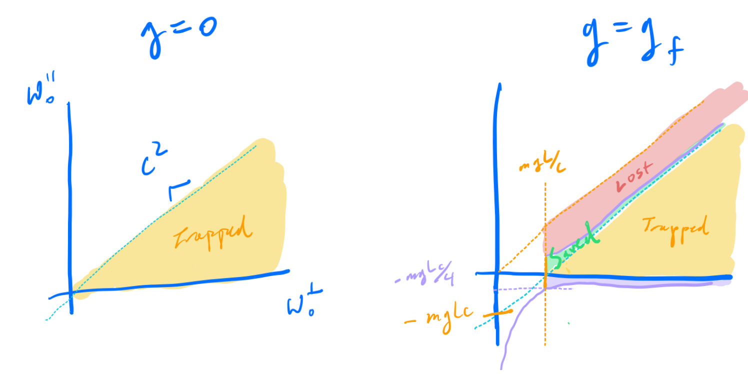
This is the adiabatic loss of parallel energy from part (g). It is possible that if a particle loses enough $W_0^{\perp}$ to stay within the shifted trapping condition, it stays trapped. While the shift in trapping condition $\Delta W_1^{\parallel}$ is constant, the adiabatic change in energy is $\Delta W_2^{\parallel} \propto 1/W_0^{\perp}$ via the $W_0^{\perp}$ dependence in z_g. Particles with

[2] What if we plotted the $W^{\parallel}(z)$ profile at different times as g changes cyclically, would we see a hysteris?

large perpendicular energy benefit less from the adiabatic absorption of parallel energy. The intersection point comes from setting $\Delta W_1^{\parallel} = \Delta W_2^{\parallel}$ which shows

$$W_0^{\perp} = \frac{mgL}{4c} < \frac{mgL}{c}$$

This condition is *always* satisfied for the new trapping condition. There is no location where all particles at a given $W_0^{\perp}$ are saved. The base case scenario is at $W_0^{\perp} = mgL/c$ where 1/4 of the particles are saved. The 'safe fraction' then decreases as $1/W_0^{\perp}$.



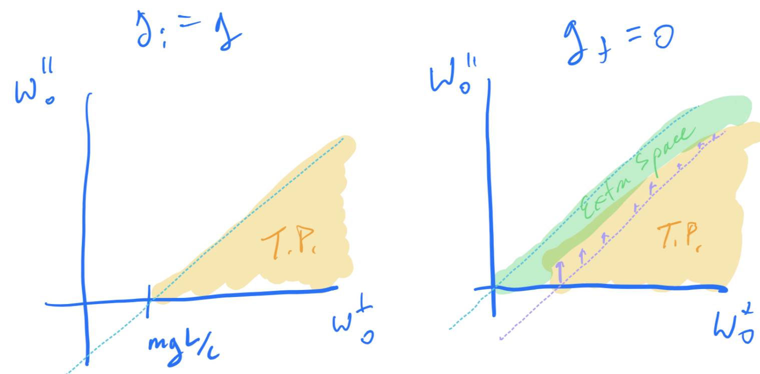
The region of de-trapped particles is bounded by $W_0^{\perp} > mgL/c$ on the right, $W_0^{\parallel}/W_0^{\perp} < c^2$ from above (the $g = 0$ trapping condition), and from below by

$$W_0^{\parallel} > c^2 \left(W_0^{\perp} - \frac{mgL}{c} \right) + \frac{(mgL)^2}{4W_0^{\perp}}$$

Therefore the corner of the de-trapped region is given by $(W_0^{\perp}, W_0^{\parallel}) = (mgL/c, mgLc/4)$.

h)

Finally we reverse the problem and slowly de-activate the field such that $g_i = g$ and $g_f = 0$. We find that



the action in Eq (9) and Eq (10) are switched. So now there is a net gain in parallel midplane energy

$$\Delta W_0^{\parallel} = \left(W_{0,f}^{\parallel} - W_{0,i}^{\parallel} \right) = \frac{1}{4} mgz_g > 0 \tag{12}$$

Also the trapping condition is relaxed. *None* of the particles are lost. Some do gain a bit of parallel energy, but this does not fully populate the new trapped particle frontier, so there is extra space to spare.