Let us consider a Spitzer Harm problem in a magnetic field. We can start from the kinetic equation

$$-\frac{e}{m_e}\left(E_{\parallel}\hat{\mathbf{b}}+\mathbf{E}_{\perp}\right)\cdot\frac{\partial f_e}{\partial \mathbf{v}}=\Omega_e\mathbf{v}\times\hat{\mathbf{b}}\cdot\frac{\partial f_e}{\partial \mathbf{v}}+C[f_e] \tag{1}$$

where we take a Lorentz collision operator

$$C[f_e]=\frac{\nu(v)}{2}\left[\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial f_e}{\partial\xi}+\frac{1}{1-\xi^2}\frac{\partial^2 f_e}{\partial\phi^2}\right]=\nu(v)L[f_e]$$

with

$$\nu(v)=\frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{Te}}{v}\right)^3$$

a)

Let us recall the Dreicier field

$$E_D=\frac{e}{\lambda_{De}^2}\ln\Lambda$$

One can write

$$\frac{eE_D}{T}=\frac{4\pi e^4 n}{T^2}\ln\Lambda=\frac{3}{\tau_{ei}v_{Te}\sqrt{2\pi}}\sim\frac{1}{\lambda_{mfp}}$$

by recognizing

$$\frac{1}{\tau_{ei}}=\frac{4}{3}\frac{e^4 n\ln\Lambda}{T_e^{3/2}}\sqrt{\frac{2\pi}{m_e}}$$

This is helpful because we can then set

$$\frac{E}{E_D}\sim\frac{J}{\left(\frac{e^2n_e\tau_{ei}}{m_e}\right)\frac{T_e}{e\lambda_{mfp}}}\sim\frac{J}{env_{Te}}\sim\frac{u_e}{v_{Te}}=\epsilon\ll 1$$

where

$$\sigma=\frac{e^2n_e\tau_{ei}}{m_e} \tag{2}$$

is the fluid conductivity. We can setup a Chapman-Enskog-Braginskii expansion for

$$\frac{E_\perp}{E_D}\sim\frac{E_\parallel}{E_D}=\epsilon\ll 1 \qquad\qquad \Omega_e\tau_{ei}\sim 1$$

Let us examine Eq (1) term by term, comparing against $\Omega_e f_0$

- parallel field

$$\frac{1}{\Omega_e f_0}\frac{eE_\parallel}{m_e}\frac{\partial f_e}{\partial v_\parallel}\sim\frac{1}{\nu_{ei}u_\parallel}\frac{eE_\parallel}{m}\sim\frac{e\lambda_{mfp}}{mvv_\parallel}E_\parallel\sim\frac{E_\parallel}{E_D}\sim\epsilon$$

- perpendicular field

$$\frac{1}{\Omega_e f_0}\frac{eE_\perp}{m_e}\frac{\partial f_e}{\partial v_\perp}\sim\frac{1}{\nu_{ei}u_\perp}\frac{eE_\perp}{m}\sim\frac{e\lambda_{mfp}}{mvv_\perp}E_\perp\sim\frac{E_\perp}{E_D}\sim\epsilon$$

- gyration

$$\frac{1}{\Omega_e f_0}\Omega_e v\frac{\partial f_e}{\partial v_\times}\sim\frac{v}{v_\times}\sim 1$$

- collisions

$$\frac{1}{\Omega_e f_0}C[f_e]\sim\frac{\nu_{ei}}{\Omega_e}\sim 1$$

This lets us write $O(1)$

$$0=-\Omega_e\frac{\partial f_0}{\partial\phi}+C[f_0] \tag{3}$$

and $O(\epsilon)$

$$-\frac{e}{m_e}\left(E_\parallel\hat{\mathbf{b}}+\mathbf{E}_\perp\right)\cdot\frac{\partial f_0}{\partial\mathbf{v}}=-\Omega_e\frac{\partial f_1}{\partial\phi}+C[f_1] \tag{4}$$

An isotropic function (not necessarily Maxwellian) $f_0(v,\xi,\phi)=f(v)$ solves Eq (3) because both $L[f]$ and ∂_ϕ are anihilated.

b)

To solve the first order problem we can take a gyroaverage.

$$\begin{aligned}0+C[\langle f_1\rangle_\phi]&=-\frac{e}{m_e}\left\langle E_\parallel\hat{\mathbf{z}}+E_\perp\left(\hat{\mathbf{x}}\cos\phi+\hat{\mathbf{y}}\sin\phi\right)\right\rangle_\phi\cdot\hat{\mathbf{v}}\,\frac{df_0}{dv}\\&=-\frac{eE_\parallel}{m_e}\xi\frac{df_0}{dv}\end{aligned}$$

where we let $\hat{v}\cdot\hat{z}=\cos\theta=\xi$. We see that not just ∂_ϕ but also $\mathbf{E}_\perp$ are eliminated by the gyroaverage. Next we recognize that the Legendre Polynomial $P_1(\xi)=\xi$ is an eigenfunction of the Lorentz operator $L[P_l]=-\frac{l(l+1)}{2}P_l$. Since the RHS has only one basis function, the LHS must match it. Thus we conclude

$$\langle f_1\rangle_\phi=\frac{eE_\parallel}{\nu(v)m_e}\frac{df_0}{dv}\xi \tag{5}$$

We may address the rest of Eq (4) using a subsidary expansion in $\delta=\frac{\nu_{ei}}{\Omega_e}\ll 1$

$$\begin{aligned}f_1&=\langle f_1\rangle_\phi+\tilde{f}_1\\&=\langle f_1\rangle_\phi+\left(\tilde{f}_1^0+\delta\tilde{f}_1^1+\delta^2\tilde{f}_1^2\right)\end{aligned}$$

Without loss of generality let us take $\mathbf{E}_\perp=\hat{\mathbf{x}}\,E_\perp\cos\phi$. Subtracting off the gyro-averaged part we find

$$-\Omega_e\frac{\partial\tilde{f}_1}{\partial\phi}+C[\tilde{f}_1]=-\frac{e}{m_e}E_\perp\cos\phi\sqrt{1-\xi^2}\frac{df_0}{dv} \tag{6}$$

where we let $\hat{v}\cdot\hat{x}=\sin\theta=\sqrt{1-\xi^2}$. Let us again examine Eq (6) term by term, comparing against $\Omega_e f_0$

- perpendicular field [1]

$$\frac{1}{\Omega_e f_0}\frac{eE_\perp}{m_e}\frac{\partial f_0}{\partial v_\perp}\sim\frac{1}{\Omega_e u_\perp}\frac{eE_\perp}{m}\sim\frac{\nu_{ei}}{\Omega_e}\frac{e\lambda_{mfp}}{mvv_\perp}E_\perp\sim\frac{\nu_{ei}}{\Omega_e}\frac{E_\perp}{E_D}\sim\delta\epsilon$$

[1] the key difference is that E/E_D brings in a factor of ν_{ei}, which pushes the base case to first order in δ. This does not occur in typical Braginskii. It is a feature of the Spitzer-Harm-Bragniskii combo.

- gyration

$$\frac{1}{\Omega_e f_0}\Omega_e \tilde{f}_1 \sim \epsilon$$

- collisions

$$\frac{C[\tilde{f}_1]}{\Omega_e f_0} \sim \frac{\nu_{ei}}{\Omega_e}\frac{\tilde{f}_1}{f_0} \sim \delta\epsilon$$

We see that to $O(\delta^0\epsilon)$

$$-\Omega_e \frac{\partial \tilde{f}_1^0}{\partial \phi} = 0$$

This means the 0th subsidiary order of $\tilde{f}_1^0$ is gyrotropic. Since $\tilde{f}_1$ already has no θ dependence, this means $\tilde{f}_1^0$ is also isotropic. [2] Therefore $C[\tilde{f}_0^1] = 0$. Thus our first order subsidiary equation $O(\delta^1\epsilon)$ is

$$-\Omega_e \frac{\partial \tilde{f}_1^1}{\partial \phi} = -\frac{e}{m_e} E_\perp \cos\phi \sqrt{1-\xi^2}\frac{df_0}{dv}$$

We can integrate this directly to find

$$\tilde{f}_1^1 = \frac{eE_\perp}{m_e\Omega_e}\frac{df_0}{dv}\sin\phi\sqrt{1-\xi^2} \tag{7}$$

Now to find the effect of collisions we should go to second order in the subsidiary expansion $O(\delta^2\epsilon)$

$$-\Omega_e \frac{\partial \tilde{f}_1^2}{\partial \phi} + C[\tilde{f}_1^1] = 0$$

This shows

$$\frac{\partial \tilde{f}_1^2}{\partial \phi} = \frac{\nu_{ei}}{\Omega_e} L[\tilde{f}_1^1] = -\frac{\nu_{ei}}{\Omega_e}\tilde{f}_1^1$$

where we recognize that $\tilde{f}_1^1$ is an eigenfunction of the Lorentz operator. [3] Consequently

$$\tilde{f}_1^2 = \frac{\nu_{ei}}{\Omega_e}\left(\frac{eE_\perp}{m_e\Omega_e}\right)\frac{df_0}{dv}\cos\phi\sqrt{1-\xi^2} \tag{8}$$

[2] Techniquely $\tilde{f}_1$ *does* have a θ dependence, since it is proportional to $\sqrt{1-\xi^2}$. A more rigorous way (found by R. Nies) to show $C[\tilde{f}_1^0] = 0$ is to argue

$$\frac{\partial}{\partial \phi}\left[\Omega_e \tilde{f}_1^1 - C[\tilde{f}_1^0]\phi - \frac{eE_\perp}{m_e}\sin\phi\sqrt{1-\xi^2}\frac{df_0}{dv}\right] = 0$$

If $C[\tilde{f}_1^0] \neq 0$, then $\tilde{f}_1^1$ would depend on ϕ non-periodically. Therefore the only way to make $\tilde{f}_1^1$ single-valued, is to enforce $C[\tilde{f}_1^0] = 0$. This is the same is saying $\tilde{f}_1$ is only allowed a coordinate dependence on θ.

[3] Although $\sqrt{1-\xi^2}$ is not orthogonal to any polynomial in the Legendre basis (1D Lorentz operator), $g = \sin\phi\sqrt{1-\xi^2}$ is an eigenfunction of the 2D (spherical) Lorentz operator

$$\begin{aligned}
L[g] &= \frac{1}{2}\left[\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi} + \frac{1}{1-\xi^2}\frac{\partial^2}{\partial \phi^2}\right] g \\
&= \left[\frac{\partial}{\partial \xi}\left(-\xi\sqrt{1-\xi^2}\right) - \frac{\sqrt{1-\xi^2}}{1-\xi^2}\right]\frac{\sin\phi}{2} \\
&= \left[\frac{2\xi^2-1}{\sqrt{1-\xi^2}} - \frac{1}{\sqrt{1-\xi^2}}\right]\frac{\sin\phi}{2} \\
&= -\sin\phi\sqrt{1-\xi^2} \\
&= -g
\end{aligned}$$

Since $\tilde{f}_1^2$ is in phase with $\mathbf{E}_\perp$, we can interpret it as the $\perp$-component. Because $\tilde{f}_1^1$ is $\pi/2$ out of phase, it can be interpreted as a $\times$-component. This matches our intution from Braginskii transport, where cross field $\perp$ depends on ν and is an order δ down from $\times$, which arises from diamagnetic flow and has nothing to do with velocity. Here this no pressure gradient, and the cross field transport is instead an $E \times B$ drift as we shall see in part (f).

c)

Let us now take f_0 to be Maxwellian

$$f_0(v) = \frac{n_e}{\pi^{3/2} v_{Te}^3} e^{-(v/v_{Te})^2}$$

It follows that

$$\frac{df_0}{dv} = -\frac{2v}{v_{Te}^2} f_0$$

Parallel current comes from $\langle f_1 \rangle_\phi$ because it is the only piece of f_1 that interacts with $E_\parallel$. We may write the kinetic moment

$$\begin{aligned} J_\parallel &= -e \int v_\parallel \langle f_1 \rangle_\phi \, d^3\mathbf{v} \\ &= -2\pi e \int v^3 \xi \langle f_1 \rangle_\phi \, d\xi \, dv \end{aligned}$$

Now expand the solution from Eq (5)

$$\langle f_1 \rangle_\phi = \frac{4\tau_{ei}}{3\sqrt{\pi}} \left(\frac{v}{v_{Te}} \right)^3 \frac{eE_\parallel}{m_e} \xi \left(-\frac{2v}{v_{Te}} \right) f_0$$

and substitute to find

$$\begin{aligned} J_\parallel &= \frac{4\tau_{ei}}{3\sqrt{\pi}} \frac{e^2 E_\parallel}{m_e} \left(2\pi \int_{-1}^{1} \xi^2 d\xi \right) \int_0^\infty v^3 \left(\frac{v}{v_{Te}} \right)^3 \left(-\frac{2v}{v_{Te}} \right) f_0 \, dv \\ &= \frac{16\tau_{ei}}{9\pi} \frac{e^2 n_e E_\parallel}{m_e} \int_0^\infty 2x^7 e^{-x^2} \, dx \\ &= \frac{16}{9\pi} \left(\frac{e^2 n_e \tau_{ei}}{m_e} \right) E_\parallel \int_0^\infty y^3 e^{-y} \, dy \\ &= \frac{32}{3\pi} \sigma E_\parallel \end{aligned}$$

The last integral is a gamma function $\Gamma(4) = 3!$. The stack of constants in parentheses is the fluid conductivity from Eq (2), which is what we set out to show.

d)

If we use the full Landau operator instead of the Lorentz pitch-angle scatter for ν_{ei}, we should also include pitch angle scattering from ν_{ee}. This would slightly decrease the σ multiplicative factor to less than $32\pi/3$ because ν_{ee} tends to isotropize the plasma, and this reduces velocity differences (which give rise to drag, and result in net current). On the other hand, if we included ion motions the conductivity would tend to increase, because $E_\parallel$ would push some ion current in the direction opposite to electrons.

e)

We are told the other moments correspond to

$$\begin{aligned}\mathbf{J} &= \mathbf{J}_\parallel + \mathbf{J}_\times + \mathbf{J}_\perp \\ &= \frac{32}{3\pi}\sigma E_\parallel \hat{\mathbf{b}} - \frac{\sigma}{\Omega_e \tau_{ei}} \mathbf{E} \times \hat{\mathbf{b}} + \frac{\sigma}{(\Omega_e \tau_{ei})^2} \mathbf{E}_\perp \\ &= \sigma \cdot \mathbf{E}\end{aligned}$$

Compared to parallel current, cross field transport is suppresed by $\delta = (\Omega_e \tau_{ei})^{-1}$ because cross field motion is inhibited by the fieldlines. In terms of a random walk arguement, the parallel step size is λ_{mfp} while the cross field stepsize is only ρ_L. This gives an heuristic explanation for

$$\frac{J_\perp}{J_\parallel} \sim \delta^2 \ll 1$$

f)

The $\times$ conductivity is independent of collision frequency

$$\sigma_\times = \frac{\sigma}{\Omega_e \tau_{ei}} = \frac{e^2 n_e}{m_e \Omega_e}$$

This $\times$ flow arises from $E \times B$ drift. Let us recall Eq (7). Using $\Omega_e = -eB/m_e$ the coefficient out front can be rewritten

$$\tilde{f}_1^1 = -\frac{E_\perp}{B}\frac{df_0}{dv}\sin\phi\sqrt{1-\xi^2}$$

In this problem both $\mathbf{E}$ and $\mathbf{B}$ are stationary and homogeneous. So $\mathbf{B}$ defines the parallel direction, the component of $\mathbf{E}$ orthogonal to $\mathbf{B}$ defines the perpendicular direction, and this uniquely species a 3rd direction for $E \times B$ drift.