

This is a quick asymptotics problem on WKB. We're given an almost-simple harmonic oscillator with slowly varying frequency

$$y'' + \Omega(t)^2 y = 0 \quad (1)$$

a)

We would like to find a leading order asymptotic. WKB says we should guess solutions of the form  $y = e^{S(t)}$ . In this case

$$\begin{aligned} y' &= S' y \\ y'' &= (S'^2 + S'') y \end{aligned}$$

Eliminating  $y$  from Eq (1) we see

$$S'^2 + S'' + \Omega^2 = 0$$

So we suppose the dominant balance <sup>1</sup>

$$S'^2 \sim -\Omega^2$$

Solving this balance yields

$$S' = \pm i\Omega$$

which we can integrate as a first order ODE

$$S(t) = \pm i \int^t \Omega(t') dt'$$

So the solution is

$$y(t) = e^{S(t)} = e^{\pm i \int^t \Omega(t') dt'}$$

A sufficient condition for this to be true, is that the dominant balance holds. So we require  $S'^2 \gg S''$ , which translates to the condition

$$\frac{\Omega'}{\Omega^2} \ll 1$$

Physically, this can be interpreted as saying the frequency  $\Omega(t)$  changes on a time scale long compared to  $\Omega^{-1}$ . We can go a step further by expanding

$$S(t) = S_0 + S_1$$

where  $S'_0 = \pm i\Omega$ . Now the expansion becomes

$$(S'_0 + S'_1)^2 + (S''_0 + S''_1) + \Omega^2 = 0$$

Subtracting off our 0th order solution leaves

$$2S'_0 S'_1 + S''_1 + S''_0 + S'_1 = 0$$

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<sup>1</sup>This comes from experience - the higher powers of lower derivatives usually end up being leading order. For example, suppose  $S = x^n$ . Then

$$S' \sim x^{n-1}$$

$$S'' \sim x^{n-2}$$

$$S'^2 \sim x^{2(n-1)}$$

Now we intuit that the dominant balance is

$$2S'_0 S'_1 \sim -S''_0$$

This gives

$$S'_1 = -\frac{S''_0}{2S'_0} = -\frac{\Omega'}{2\Omega}$$

Therefore

$$S_1(t) = -\frac{1}{2} \ln \left( \frac{\Omega(t)}{\Omega(0)} \right)$$

Thus the leading order solution is

$$\begin{aligned} y(t) = e^{S_0 + S_1} &\sim \frac{1}{\sqrt{\Omega}} e^{\pm i \int^t \Omega(t') dt'} \\ &= \frac{1}{\sqrt{\Omega_t / \Omega_0}} e^{\pm i \int^t \Omega_{t'} dt'} \end{aligned}$$

This is a sinusoid with slowly changing amplitude.

b)

In physics there is a general concept of action

$$J = \oint p dq$$

which describes the adiabatic invariance of quasi-periodic behavior. In the limit when  $\Omega$  does not change at all, our problem admits an exact time invariance. To see this multiply Eq (1) by  $y'$

$$y' y'' + \Omega^2 y y' = 0$$

and integrate to see

$$\frac{d}{dt} \left[ \frac{y'^2}{2} + \Omega^2 \frac{y^2}{2} \right] = 0 \quad (2)$$

This is energy conservation. We can call the (almost) conserved quantity  $E$ . Leveraging this quasi-constant to write

$$y' = \sqrt{2E - (\Omega y)^2}$$

our action integral becomes

$$\begin{aligned} J &= \oint y' dy = \sqrt{2E} \oint \sqrt{1 - \frac{(\Omega y)^2}{2E}} dy \\ &= \frac{2E}{\Omega} \left( \oint \sqrt{1 - x^2} dx \right) \end{aligned}$$

The cyclic integral is a geometric constant.<sup>2</sup> There the ratio of quasi-constants  $E$  which is slowly varying and  $\Omega$  which is slowly varying, can come out to be a constant which varies even more slowly. To quantify this we can expand Eq (2), putting back  $\Omega = \Omega(t)$  to say

$$E' = -\Omega' \Omega y^2$$

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<sup>2</sup>Let  $x = \sin \theta$ , then

$$\oint \sqrt{1 - x^2} dx = \oint \cos^2 \theta d\theta = \pi$$

This lets us Taylor expand

$$\begin{aligned}
J &= 2\pi \left( \frac{E}{\Omega} \right) = 2\pi \left[ \frac{E_0 - (\Omega' \Omega y^2)t + \dots}{\Omega_0 + \Omega' t + \dots} \right] \\
&= 2\pi \frac{E_0}{\Omega_0} \left( 1 - \frac{\Omega' \Omega y^2}{E_0} t + \dots \right) \left( 1 - \frac{\Omega'}{\Omega} t + \dots \right) \\
&= 2\pi \frac{E_0}{\Omega_0} \left[ 1 - \frac{\Omega'}{\Omega^2} t \left( \frac{\Omega^2 y^2}{E_0} + \Omega \right) + \dots \right]
\end{aligned}$$

We see again that, if  $S''/S'^2 = \Omega'/\Omega^2 \ll 1$ , then time variation is suppressed asymptotically.