We consider a system governed *strictly* by Vlasov and Poisson equations

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s - \frac{q_s}{m_s} \nabla \varphi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0 \tag{1}$$

$$\nabla^2 \varphi = -4\pi \sum_s q_s \int f_s \, d^3\mathbf{v} \tag{2}$$

## a)

Let us define the total energy

$$E = \int U \, d^3\mathbf{r}$$

where

$$U = \frac{|\nabla \varphi|^2}{8\pi} + \sum_s \int \left( \frac{1}{2} m_s v^2 \right) f_s \, d^3\mathbf{v}$$

Here $U$ is a local energy density, where the first term is potential energy stored in the field, while the second term is kinetic energy of the distributions $f_s$.

## b)

We would like to show that the total energy is conserved $dE/dt = 0$. Let us compute

$$\frac{dE}{dt} = \int \frac{\partial U}{\partial t} \, d^3\mathbf{r}$$

Since the boundary for the volume integral is all space, it has no time derivative. Since $v$ is a phase space coordinate rather than a kinematic velocity, it has no time derivative. Thus the second term is just

$$\partial_t U_k = \sum_s \int \left( \frac{1}{2} m_s v^2 \right) \frac{\partial f_s}{\partial t} \, d^3\mathbf{v}$$

For the first time, we should leverage the integral over all space to integrate-by-parts the gradients

$$\begin{aligned}
\frac{\partial}{\partial t} \int U_p \, d^3\mathbf{r} &= \frac{\partial}{\partial t} \left( \frac{1}{8\pi} \int |\nabla \varphi|^2 \, d^3\mathbf{r} \right) \\
&= \frac{1}{4\pi} \int (\nabla \dot{\varphi}) \cdot (\nabla \varphi) \, d^3\mathbf{r} \\
&= -\frac{1}{4\pi} \int \phi \nabla^2 \dot{\varphi} \, d^3\mathbf{r} + \frac{1}{4\pi} \int \nabla \cdot (\varphi \nabla \dot{\varphi}) \, d^3\mathbf{r}
\end{aligned}$$

The boundary term vanishes because

$$\int \nabla \cdot (\varphi \nabla \dot{\varphi}) \, d^3\mathbf{r} = \oint_\partial \varphi \nabla \dot{\varphi} \, d^2\mathbf{r} = 0$$

since $\varphi = 0$ on the boundary at infinity. The remaining term can be expressed using Eq (2) as

$$\frac{\partial}{\partial t} \int U_p \, d^3\mathbf{r} = \int \phi \frac{\partial}{\partial t} \left( \frac{-\nabla^2 \varphi}{4\pi} \right) d^3\mathbf{r} = \int \left( \sum_s q_s \phi \int \frac{\partial f_s}{\partial t} \, d^3\mathbf{v} \right) d^3\mathbf{r}$$

So all together

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\int (U_p + U_k)\, d^3\mathbf{r} = \int d^3\mathbf{r}\sum_s \int d^3\mathbf{v}\left[q_s\varphi + \frac{1}{2}m_s v^2\right]\frac{\partial f_s}{\partial t}$$

Let us recognize that the term in square brackets in single particle energy $u = q_s\varphi + \frac{1}{2}m_s v^2$. Although it is not conserved, we can make use of the grouping for physical intuition. Let us use Vlasov's Eq 1 to substitute

$$\frac{dE}{dt} = -\sum_s \int d^3\mathbf{r}\int d^3\mathbf{v}\left[q_s\varphi + \frac{1}{2}m_s v^2\right]\left(\mathbf{v}\cdot\nabla f_s - \frac{q_s}{m_s}\nabla\varphi\cdot\partial_{\mathbf{v}} f_s\right)$$

We would like to isolate $f_s$, so let us integrate by parts the first term in parentheses in space and the second term in velocity

$$\frac{dE}{dt} = \sum_s \int d^3\mathbf{r}\int d^3\mathbf{v}\, f_s\left(\mathbf{v}\cdot\nabla - \frac{q_s}{m_s}\nabla\varphi\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left[q_s\varphi + \frac{1}{2}m_s v^2\right]$$

where we note $\mathbf{v}$ is independent of $\nabla$, and $-(q_s/m_s)\nabla\varphi$ is independent of $\partial_{\mathbf{v}}$. But these operators each anihilate one of the terms in single particle energy $u$. So we find

$$\frac{dE}{dt} = \sum_s \int d^3\mathbf{r}\int d^3\mathbf{v}\, f_s\left(q_s\mathbf{v}\cdot\nabla\varphi - q_s\nabla\varphi\cdot\mathbf{v}\right) = 0$$

where the integrand on the RHS vanishes identically. We can interpret the final expression physically

$$P = (-q_s\nabla\varphi)\cdot\mathbf{v} = \mathbf{F}_E\cdot\mathbf{v}$$

is the power transfered from field to particle. Our algebraic manipulation is saying the the rate of energy transfer from particle into field exactly matches that from field into particles. Thus net power transfer is 0, and global energy is conserved.

## c)

Now we search for a local energy conservation of the form

$$\frac{\partial U}{\partial t} + \nabla\cdot(\mathbf{F} + \mathbf{P}) = 0$$

where

$$\mathbf{F} = \sum_s \int \left(\frac{1}{2}m_s v^2\right)\mathbf{v} f_s\, d^3\mathbf{v}$$

We can interpret $\mathbf{F}$ as the kinetic heat flux summed over all species. It is a 3rd order moment on the distributions $f_s$. If there is a local change in energy $\partial U/\partial t$ then that energy is carried away either by particles as kinetic heat flux $\mathbf{F}$, or by the field as "electrostatic momentum flux" $\mathbf{P}$.

## d)

We would like to find an expression for $\mathbf{P}$. This is tricky, because in electrostatics there is no magnetic field $\mathbf{B} = 0$. Hence there is no EM radiation or Poynting flux. Nonetheless, we can compute the quantity

$$\begin{aligned}
-\nabla\cdot\mathbf{P} &= \nabla\cdot\mathbf{F} + \frac{\partial U}{\partial t} \\
&= \sum_s \int \nabla\cdot\left(\frac{1}{2}m_s v^2\mathbf{v} f_s\right) d^3\mathbf{v} + \sum_s \int \frac{\partial}{\partial t}\left(\frac{1}{2}m_s v^2 f_s\right) d^3\mathbf{v} + \frac{\partial}{\partial t}\frac{|\nabla\varphi|^2}{8\pi}
\end{aligned}$$

Using Vlasov's equation the first two terms can be combined

$$\begin{aligned}\nabla(f_s\mathbf{v}) + \frac{\partial f_s}{\partial t} &= \frac{q_s}{m_s}\nabla\varphi\cdot\frac{\partial f_s}{\partial \mathbf{v}} \\ &= \frac{q_s}{m_s}\left[\nabla\cdot\left(\varphi\frac{\partial f_s}{\partial \mathbf{v}}\right) - \varphi\nabla\cdot\frac{\partial f_s}{\partial \mathbf{v}}\right]\end{aligned}$$

While the field term can be expressed

$$\begin{aligned}\frac{\partial}{\partial t}\frac{|\nabla\varphi|^2}{8\pi} &= \frac{1}{4\pi}\nabla\varphi\cdot\nabla\left(\frac{\partial\varphi}{\partial t}\right) \\ &= \frac{1}{4\pi}\left[\nabla\cdot\left(\varphi\nabla\frac{\partial\varphi}{\partial t}\right) - \varphi\nabla^2\frac{\partial\varphi}{\partial t}\right]\end{aligned}$$

In both steps we anticipated an integration by parts. The enables us to write

$$-\nabla\cdot\mathbf{P} = \nabla\cdot\left[\sum_s\int\varphi\left(\frac{1}{2}q_s v^2\right)\frac{\partial f_s}{\partial \mathbf{v}}\,d^3\mathbf{v} + \varphi\nabla\left(\frac{1}{4\pi}\frac{\partial\varphi}{\partial t}\right)\right] - \varphi\left\{\sum_s\int\left(\frac{1}{2}q_s v^2\right)\nabla\cdot\frac{\partial f_s}{\partial \mathbf{v}}\,d^3\mathbf{v} + \frac{\partial}{\partial t}\frac{\nabla^2\varphi}{4\pi}\right\}$$

If the term in curly brackets can be shown to vanish, then the term in square brackets must be $-\mathbf{P}$. To show that this boundary term vanishes, we can start by applying Poisson's equation

$$\frac{\partial}{\partial t}\frac{\nabla^2\varphi}{4\pi} = -\sum_s q_s\int\frac{\partial f_s}{\partial t}\,d^3\mathbf{v}$$

while

$$\sum_s\int\left(\frac{1}{2}q_s v^2\right)\nabla\cdot\frac{\partial f_s}{\partial \mathbf{v}}\,d^3\mathbf{v} = -\sum_s q_s\int\mathbf{v}\cdot\nabla f_s\,d^3\mathbf{v}$$

Next we sum the two and apply Vlasov's equation

$$\{\ldots\} = \sum_s\frac{q_s^2}{m_s}\int\nabla\varphi\cdot\frac{\partial f_s}{\partial \mathbf{v}}\,d^3\mathbf{v} = \sum_s\frac{q_s^2}{m_s}\int\frac{\partial}{\partial \mathbf{v}}\cdot(f_s\nabla\varphi)\,d^3\mathbf{v} = 0$$

where the integral of the gradient must vanish because $f_s = 0$ at infinite velocities. Therefore we find

$$\begin{aligned}\mathbf{P} &= -\sum_s\int\varphi\left(\frac{1}{2}q_s v^2\right)\frac{\partial f_s}{\partial \mathbf{v}}\,d^3\mathbf{v} + \frac{\varphi}{4\pi}\frac{\partial}{\partial t}(-\nabla\varphi) \\ &= \varphi\sum_s\int q_s\mathbf{v}f_s\,d^3\mathbf{v} + \frac{\varphi}{4\pi}\frac{\partial\mathbf{E}}{\partial t} \\ &= \frac{c}{4\pi}\varphi\left[\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right]\end{aligned}$$

## (speculative discussion)

Let us note that the form for $\mathbf{P}$ strongly resembles Ampere's law.

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

If we had full Maxwell's equations, we would find

$$
\begin{aligned}
\nabla \cdot \mathbf{P} &= \frac{c}{4\pi} \nabla \cdot (\varphi \nabla \times \mathbf{B}) \\
&= \frac{c}{4\pi} \nabla \cdot (\mathbf{B} \times \nabla \varphi) \\
&= \nabla \cdot \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right)
\end{aligned}
$$

which recovers the Poynting vector. In the electrostatic *approximation of Maxwell's equations* $\mathbf{B} = 0$ implies $\mathbf{P} = 0$. This can be understand by saying electrostatic physics applies only in the regime where particles move slow enough that radiation is neglible

$$
F = q \left( E + \frac{v}{c} B \right) \approx qE \tag{3}
$$

However in our strict adherence to electrostatic Vlasov-Poisson, there is action at a distance. Solving Poisson's equation says

$$
\varphi(\mathbf{r}) = \int \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left( \sum_s q_s \int f_s \, d^3\mathbf{v} \right) d^3\mathbf{r}'
$$

The scalar field $\varphi$ knows about changes in $f_s$ over all space *instantaneously*, but this is not a problem because there is no relativity. We can invert the $v/c \ll 1$ condition from Eq (3) as formally sending $c \to \infty$. So in a paradoxical but self-consistent sense we have dropped relativity but nonetheless recovered radiation and local energy conservation.