Alfven Resonance (2021 - II.3) **Tony M. Qian : 4 May 2021**

We consider a system of electrons and ions with background field $\mathbf{B}_0$ pointing along $\hat{z}$. The plasma is homogeneous along y and z, but there is both ∇n and ∇B in the $\hat{x}$ direction. Let us start with the cold fluid dielectric

$$\epsilon = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

We can recall [1]

$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}$$

$$D = \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}\frac{\Omega_s}{\omega}$$

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

We consider waves that propagate in $k_z > 0$, but the amplitude varies along the inhomogeneity

$$\tilde{\mathbf{E}} = \mathbf{E}(x)e^{i(k_z z - \omega t)}$$

We will take the MHD approximation $\delta = \omega/\Omega_i \ll 1$. This makes

$$S = 1 + \frac{\omega_{pe}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{\Omega_i^2} \approx \frac{\omega_{pi}^2}{\Omega_i^2}$$

since $|\Omega_e| \gg \Omega_i$. We also have

$$\begin{aligned} D &= -\sum_s \frac{\Omega_s}{\omega}\frac{\omega_{ps}^2}{\Omega_s^2}\left[1 + \frac{\omega^2}{\Omega_s^2} + \dots\right] \\ &= -\frac{1}{\omega}\sum_s \frac{\omega_{ps}^2}{\Omega_s} - \sum_s \frac{\omega_{ps}^2}{\Omega_s^2}\frac{\omega}{\Omega_s} \end{aligned}$$

The first term vanishes because of quasineutrality. [2] Thus

$$D = -\frac{\omega_{pi}^2}{\Omega_i^2}\frac{\omega}{\Omega_i} - \frac{\omega_{pe}^2}{\Omega_e^2}\frac{\omega}{\Omega_e} \approx -S\frac{\omega}{\Omega_i}$$

We are told in the problem statement that $S \gg 1$. This means that $D = -\delta S$ is the product of a large and small number, so its actual size is yet undetermined.

---

[1] It may help to remember the 'sum' and 'difference' mnemonic

$$S = \frac{R+L}{2} \qquad R, L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega \pm \Omega_s)} \qquad D = \frac{R-L}{2}$$

where 'right' and 'left' correspond to handedness of magnetic gyration. Electrons are right, $\Omega_e < 0$, so resonance happens iwth the $(+)$. Ions are left, $\Omega_i > 0$, so resonance corresponds the $(-)$.

[2]

$$\sum_s \frac{\omega_{ps}^2}{\Omega_s} = \sum_s \left(\frac{4\pi q_s^2 n_s}{m_s}\right)\left(\frac{m_s c}{q_s B}\right) = \frac{4\pi c}{B}\sum_s q_s n_s = 0$$

a) Alfven Waves

What does it mean for $S \gg 1$? Consider

$$\frac{\omega_{pi}^2}{\Omega_i^2} = \left(\frac{4\pi e^2 n_i}{m_i}\right)\left(\frac{m_i^2}{e^2 B_0^2}\right) = \frac{4\pi m_i n_i}{B_0^2} = \left(\frac{c}{v_A}\right)^2$$

This quantity is defined γ_A in the notes. It is like a refractive index $n = ck/\omega$ for Alfven waves. $S \approx \gamma_A$ is large because the Alfven velocity is small compared to the spped of light. Even though electrostatic oscillations become stationary as $\omega \to 0$, the plasma is responding through magnetic vibrations. This is because particles, tied to the magnetic field lines, see an oscillating electric field in the co-moving frame. This give rise to polarization current.

b) Dispersion Relation

Starting from Maxwell's equations in matter

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

We can use the constitutive reltaions $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{H} = \frac{\mathbf{B}}{\mu} \approx \mathbf{B}$ to write

$$-\nabla \times (\nabla \times \mathbf{E}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\epsilon \mathbf{E} = \frac{4\pi}{c^2}\frac{\partial \mathbf{J}}{\partial t}$$

Since we're interested in waves, we take the homogeneous solution. [3] We're also told that E_z is neglible, because P is large. [4] This gives

$$\left[-\begin{pmatrix} \frac{d^2}{dx^2} & 0 & ik_z\frac{d}{dx} \\ 0 & 0 & 0 \\ ik_z\frac{d}{dx} & 0 & -k_z^2 \end{pmatrix} + \left(\frac{d^2}{dx^2} - k_z^2\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\omega^2}{c^2}\begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}\right]\begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can convert to dimensionless refractive index $N = \frac{ck}{\omega}$

$$\begin{pmatrix} S - N_z^2 & -iD & N_z\frac{c}{i\omega}\frac{d}{dx} \\ iD & S - N_z^2 + \frac{c^2}{\omega^2}\frac{d^2}{dx^2} & 0 \\ N_z\frac{c}{i\omega}\frac{d}{dx} & 0 & P + \frac{c^2}{\omega^2}\frac{d^2}{dx^2} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The top two rows show

$$(S - N_z^2)E_x = iDE_y$$

[3] This is just

$$D_{EM} = |N\rangle\langle N| - \langle N|N\rangle + \epsilon = 0$$

except we used $N = \frac{c}{\omega}\left(\frac{1}{i}\frac{d}{dx}, 0, k_z\right)$.

[4] This makes sense because

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

diverges for $\omega \to 0$. But the dispersion relation (3rd row) gives

$$PE_z = N_\perp^2 E_z - N_\parallel N_\perp E_x$$

The RHS in finite. To balance P, E_z must be small (but not 0!). This lets us neglect the parallel component and focus on the 2x2 $E_\perp$ matrix going forward.

$$-iDE_x = \left(\frac{c^2}{\omega^2}\frac{d^2}{dx^2} + S - N_z^2\right) E_y$$

and eliminating E_x yields the dispersion relation we desire

$$\left(\frac{c^2}{\omega^2}\frac{d^2}{dx^2} + S - N_z^2 - \frac{D^2}{S - N_z^2}\right) E_y = 0 \tag{1}$$

c) Alfven Resonance

Now let us give more detail to the inhomogeneity. Suppose

$$S = \left(1 + \frac{x}{L}\right) N_z^2$$

We see that as $x \to 0$ we have $S \to N_z^2$, which creates a resonance in Eq (1). This is called the Alfven resonance. It is related to the L-mode and ion cyclotron resonance, but occurs for $\omega < \Omega_i$ and both $N_\parallel$ and $N_\perp$ components play a role. We would like to study what happens when the resonance is approached from above and below ($x > 0$ and $x < 0$). The distance to resonance is parametrized by

$$S - N_z^2 = \frac{x}{L} N_z^2$$

Also

$$D \approx -\frac{\omega}{\Omega_i} S = -\delta \left(1 + \frac{x}{L}\right) N_z^2$$

Let $\rho = x/L$ be a dimensionless position. Considering $E_y \neq 0$, Eq (1) becomes

$$\left(\frac{c}{\omega L}\right)^2 \frac{d^2}{d\rho^2} + \rho N_z^2 - \frac{D^2}{\rho N_z^2} = 0$$

This gives

$$0 = \left(\rho N_z^2\right) \frac{d^2}{d\rho^2} + \frac{\rho^2 N_z^4 - D^2}{\left(\frac{c}{\omega L}\right)^2}$$

We should substitute D because it has a (suppressed) x dependence

$$0 = \rho \frac{d^2}{d\rho^2} + \frac{\rho^2 - \delta^2 (1 + \rho)^2}{\left(\frac{c/\omega L}{N_z}\right)^2}$$

Next we make an asymptotic approximation. Since we're interested in the resonance at $x \to 0$, it is natural to set $x \ll L$ such that $1 + \rho \approx 1$. Then

$$0 = \rho \frac{d^2}{d\rho^2} + \frac{(\rho/\delta)^2 - 1}{\left(\frac{c/\omega L}{\delta N_z}\right)^2}$$

This motivates defining a stretched parameter

$$\xi = \frac{\rho}{\delta} = \frac{x}{L}\frac{\Omega_i}{\omega}$$

The Alfven resonance occurs in an asymptotic boundary layer. This is set by the the density and magnetic field gradients, which S accesses through ω_{pi} and Ω_i respectively. Now scaling the entire equation by δ we find

$$0 = \xi \frac{d^2}{d\xi^2} + \delta^3 (k_z L)^2 \left(\xi^2 - 1\right)$$

where we have expanded $\left(\frac{\omega L}{c}\right) N_z = k_z L$. Defining

$$\alpha^2 = \frac{1}{2\delta^3 (k_z L)^2}$$

yields

$$\left(\xi \frac{d^2}{d\xi^2} + \frac{\xi^2 - 1}{2\alpha^2}\right) E_y = 0 \tag{2}$$

which is what we set out to show. We will use this as the governing equation going forward. Let us take a geometric optics approximation

$$\frac{1}{i}\frac{d}{d\xi} = k_\xi$$

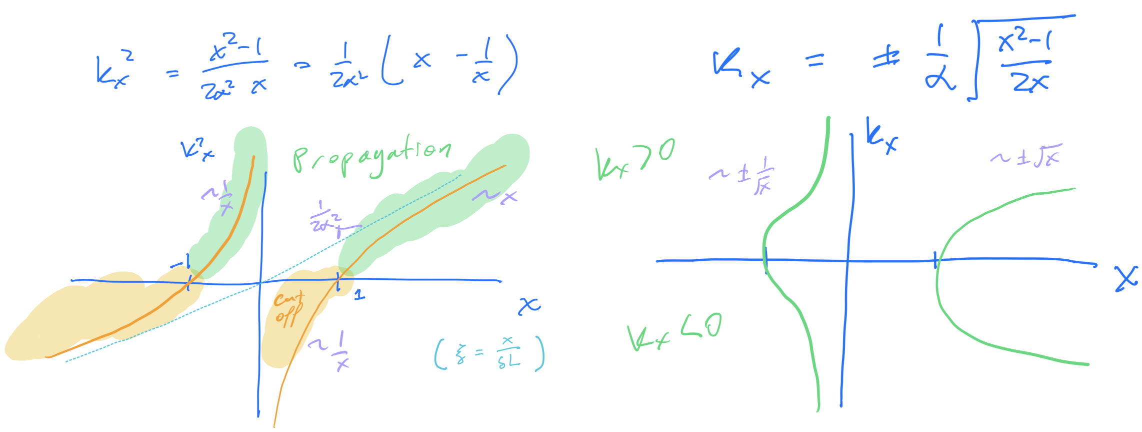
This sets

$$-k_\xi^2 \xi + \frac{\xi^2 - 1}{2\alpha^2} = 0$$

which we can solve to find

$$k_\xi^2 = \frac{1}{2\alpha^2}\left(\xi - \frac{1}{\xi}\right)$$

In the k_ξ^2 plot (left) we see two regions of propagation separated by a cut-off and a resonance. The cut-off layer has a width $\Delta x = \frac{\omega}{\Omega_i} L$. This suggests for sufficiently small δ we might find mode conversion. In the $\pm k_\xi$ plot (right) we see two branches, which each experience cut off (reflection) at $\rho = \pm 1$. The algebraic



form for these branches can be found by solving the inverse equation [5]

$$\xi^2 - 2(k_\xi \alpha)^2 \xi - 1 = 0$$

which shows

$$\begin{aligned} \xi &= (k_\xi \alpha)^2 \pm \sqrt{1 + (k_\xi \alpha)^4} \\ &= \bar{\xi} \pm \Delta \end{aligned}$$

Here we should emphasize that both pieces are functions of k_ξ

$$\bar{\xi}(k_\xi) = (k_\xi \alpha)^2$$

$$\Delta\xi(k_\xi) = \sqrt{1 + (k_\xi \alpha)^4}$$

[5]the problem statement asks to plot both $k(\xi)$ and $\xi(k)$, but I will defer the latter to part (d).

d) action flow

We would like to find the x-direction group velocity

$$v_{g,x} = \frac{\partial \omega}{\partial k_\xi}$$

Starting from the dispersion function

$$D(\omega, k_\xi) = -k_\xi^2 \xi + \frac{\xi^2 - 1}{2\alpha^2} = 0$$

We can isolate ω by solving the frequency dependence in $\alpha^{-2} = 2(k_z L)^2 \delta^3$. [6] This yields

$$(k_z L)^2 \left(\frac{\omega}{\Omega_i}\right)^3 = \frac{1}{2\alpha^2} = \frac{k_\xi^2 \xi}{\xi^2 - 1}$$

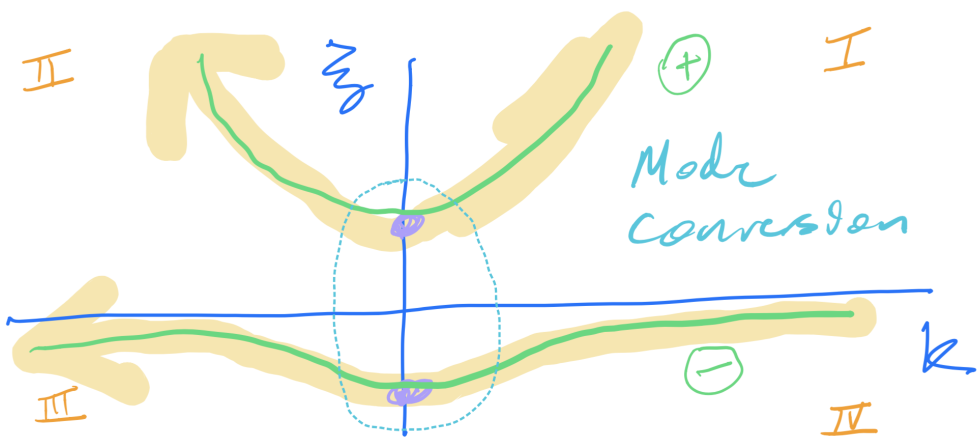
or

$$\omega = k_\xi^{2/3} \Omega_i \left(\frac{\xi}{(k_z L)^2 (\xi^2 - 1)}\right)^{1/3}$$

From here taking asimple derivitive shows

$$v_{g,\xi} = \frac{2}{3} \Omega_i \left(\frac{\xi}{k_\xi}\right)^{1/3} \left[(k_z L)^2 (\xi^2 - 1)\right]^{-1/3}$$

Thus the direction of progation depends on the sign of ξ / k_ξ as well as the sign of $(\xi^2 - 1)$. Since $\xi^2 \geq 1$ for the upper branch and $\xi^2 < 1$ for the lower branch, we find that both both branches propagate toward $-k_\xi$. We can see that the $(+)$ branch propagates from above the boundary layer $(\xi > 1)$ in quadrant I, hits the



cut off at $\xi = 1$, and reflects into quadrant II. Below the boundary layer $(\xi < 0)$, the $(-)$ branch propagates away from the resonance $(k < 0)$ in quadrant III, reflects at $\xi = -1$, and propagates toward the cut off in quardrant IV. We can interpret this to mean that the action propagates in opposite directions for the

[6] this path was found by T. Rubin.

two branches. [7] Mode conversion occurs if part of wave incoming from I reflects into II, while part jumps branches continuing toward resonance in IV. To identify the branches it is helpful to consider the ratio

$$\beta = \frac{S - N_z^2}{D} = \frac{\rho}{\delta(\rho - 1)}$$

For $\rho > 1$ we see $\beta \sim \delta^{-1} \gg 1$. This $(+)$ branch corresponds to the fast CAW. [8] For $\rho \to 0^-$ we see that $\beta \to 0^+ \ll 1$. [9] This $(-)$ branch corresponds to the SAW, and it is the one which ultimately rises to the Alfven resonance.

e) mode conversion

Now we're given a Schrodinger like equation

$$i\frac{d}{dk_\xi}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} s & -i \\ i & -s \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

where $s = (k_\xi \alpha)^2$ as before, and $g_{1,2} \propto e^{-i\theta_{1,2}(k_\xi)}$. [10] Let us define $\zeta_{1,2} = \frac{d\theta_{1,2}}{dk_\xi}$ We see that the Schrodinger equation is an eigenvalue equation

$$H\mathbf{g} = i\frac{d\mathbf{g}}{dk_\xi} = i\begin{pmatrix} -i\theta_1' g_1 \\ -i\theta_2' g_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 g_1 \\ \zeta_2 g_2 \end{pmatrix}$$

So finding $\zeta_{1,2}$ is equivalent to finding the eigenvalues.

$$0 = \det\begin{pmatrix} s - \lambda & -i \\ i & -(s+\lambda) \end{pmatrix} = \lambda^2 - s^2 - 1$$

This means the eigenvalues are

$$\lambda_\pm = \pm\sqrt{1 + s^2}$$

which correspond to the values $\Delta(k_\xi)$ from our dispersion relation $\xi = \bar{\xi} \pm \Delta$. Without loss of generality, let us assign

$$\zeta_1 = \lambda_+ = \sqrt{1 + s^2}$$

$$\zeta_2 = \lambda_- = -\sqrt{1 + s^2}$$

[7] Recall

$$\frac{\partial I}{\partial t} + \nabla \cdot (\mathbf{v_g} I) = 2\gamma I$$

[8] For cold fluid waves there is no slow CAW at low frequency. because $C_S \to 0$. Both SAW and slow CAW can never propagate purely $k \perp B$, there always needs to be some parallel component, but fast CAW can. At $\theta = \pi/2$ the fast CAW is the low frequency limit of X-modes. At $\theta = 0$ the fast CAW is the low frequency limit of R-modes, and the SAW is the low frequncy limit of L-modes.

[9] Let us note that $\beta \approx 1$ when $\rho \sim -\delta$ $(\xi = -1)$, and we have

$$S - D = L \approx N_z^2$$

This is called "the left-hand cutoff". How can this be connected to the L-mode branch $N_\parallel^2 - L = 0$ from Appleton-Hartree, which has no cutoff?

[10] For background consider

$$g_1(k_\xi) = e^{i\Theta(k_\xi)} \int E_y(x) e^{ik_x x} dx$$

where Θ is the anti-derivative of $\bar{x}$

$$\Theta(k_\xi) = \int \bar{x}(k_\xi)\, dk_\xi$$

Therefore the frequency gap is

$$|\zeta_1 - \zeta_2| = 2|\lambda| = 2\sqrt{1+s^2}$$

Since $\zeta_{1,2} = \lambda_\pm = \pm\Delta$, we can interpret this to be the separation of modes in k_ξ-space. We can speculate that mode conversation is more likely when this separation is smaller. Now it remains only to find the eigenvectors

$$\begin{pmatrix} sg_1 - ig_2 \\ ig_1 - sg_2 \end{pmatrix} = H\mathbf{g} = \begin{pmatrix} \zeta_1 g_1 \\ \zeta_2 g_2 \end{pmatrix}$$

After some algebra we find the orthonormal eigenvectors associated with these eigenvalues are

$$h_+ = \begin{pmatrix} -i\sqrt{\frac{1+\kappa}{2}} \\ \sqrt{\frac{1-\kappa}{2}} \end{pmatrix} = \begin{pmatrix} -iu_+ \\ u_- \end{pmatrix}$$

$$h_- = \begin{pmatrix} i\sqrt{\frac{1-\kappa}{2}} \\ \sqrt{\frac{1+\kappa}{2}} \end{pmatrix} = \begin{pmatrix} iu_- \\ u_+ \end{pmatrix}$$

with $\kappa = s/\Delta = \frac{s}{\sqrt{1+s^2}}$ and $u_\pm = \sqrt{(1\pm\kappa)/2}$. Let us study the following limits

$$\begin{array}{rrr} s \to 0 & s \to \infty & s \to -\infty \\ \kappa \to 0 & \kappa \to 1 & \kappa \to -1 \\ u_+ \to 1/\sqrt{2} & u_+ \to 1 & u_+ \to 0 \\ u_- \to 1/\sqrt{2} & u_- \to 0 & u_- \to 1 \end{array}$$

In our system [11]

$$s = (k_\xi \alpha)^2 = \frac{1}{2\delta}\left(\frac{N_\perp}{N_\parallel}\right)^2 \geq 0$$

We see that for large α ($s \to +\infty$) the incoming wave stays on the same branch. But for small α ($s \to 0$) the g_1 and g_2 amplitudes are equal. Another way to look at this is that in the limit $s \to 0$ the matrix becomes anti-symmetric

$$M_{s\to 0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

This means g_1 and g_2 interchange places. So if we interpret the initial state to be $|i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for example, this corresponds to a 100% probability of jumping modes. Therefore mode conversion is more likely at smaller values of α i.e. the $N_\parallel$ is low, so the wave spends more time at the boundary layer $S(x=0)$. Conversely if α is large, $N_\parallel$ is large, and the wave passes right through the boundary layer without much mode conversion.

[11] Note

$$\xi = \frac{\rho}{\delta} = \frac{x}{\delta L}$$

implies

$$k_\xi = \frac{1}{i}\frac{d}{d\xi} = \frac{\delta L}{i}\frac{d}{dx} = \delta L k_x$$

which shows

$$s = (k_\xi \alpha)^2 = \frac{k_\xi^2}{2\delta^3 (k_z L)^2} = \frac{1}{2\delta^3 L^2}\left(\frac{k_\xi}{k_z}\right)^2 = \frac{1}{2\delta}\left(\frac{k_x}{k_z}\right)^2 = \frac{1}{2\delta}\left(\frac{N_\perp}{N_\parallel}\right)^2$$