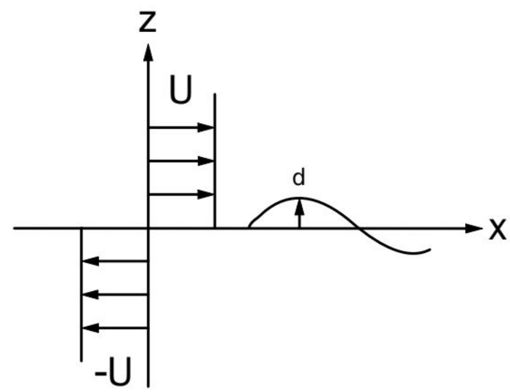
I.7 Kelvin-Helmholtz Instability [40 pts]

In this problem, you will be asked to derive minimum magnetic field to stabilize Kelvin-Helmholtz instability in MHD. Suppose that an incompressible, ideal fluid has a flow shear at a sharp boundary at $z = 0$ as shown in the figure where a proper frame is chosen so that $\mathbf{U} = (U, 0, 0)$ at $z > 0$ while $\mathbf{U} = (-U, 0, 0)$ at $z < 0$.



(a) [8 pts] Assuming the perturbation velocity, $\mathbf{v}(x, z)$, is irrotational and in the (x, z) plane, so that a velocity potential, $\phi(x, z)$, is allowed to represent $\mathbf{v}$:

$$\mathbf{v} = \nabla\phi. \tag{2}$$

When $\phi = \phi_1(z)e^{ik(x-ct)}$ at $z > 0$ and $\phi = \phi_2(z)e^{ik(x-ct)}$ at $z < 0$, derive equations for ϕ_1 and ϕ_2 and find their solutions. Here, k is real and c is complex. Let $\alpha \equiv \phi_1(z \to +0)$ and $\beta \equiv \phi_2(z \to -0)$.

(b) [8 pts] As shown in the figure, suppose that the boundary perturbation is given by $\xi = de^{ik(x-ct)}$ where d is the amplitude. Use the kinematic boundary condition of

$$v_z = \frac{D\xi}{Dt} \tag{3}$$

for both $z \to +0$ and $z \to -0$ to express α and β, respectively, in terms of d. Here D/Dt denotes total derivative.

(c) [8 pts] Express perturbed pressure, $p_1[\equiv p(z \to +0)]$, in terms of α and $p_2[\equiv p(z \to -0)]$, in terms of β, respectively. Use the dynamic boundary condition,

$$p_1 = p_2, \tag{4}$$

to derive dispersion relation for Kelvin-Helmholtz instability. When is it unstable?

(d) [8 pts] Impose a uniform magnetic field B_0 along the flow direction. Express the perturbed magnetic field in the x direction, B_x, in terms of ϕ for $z > 0$ and for $z < 0$, respectively. [Vector identity: $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$.]

a) $\nabla \times U = 0$ so $U = \nabla \phi(x,z)$ See Hantao's Lec 9 Notes

$\nabla \cdot U = 0$ so $\nabla^2 \phi = 0$

$$\nabla^2 \phi = 0 = \frac{\partial^2 \phi(z)}{\partial x^2} - k^2 \phi(z) = 0$$

$$\phi(z) = \left(\beta e^{|k|z} + \alpha e^{-|k|z}\right) e^{ik(x-ct)}$$

b) No divergence allowed: $\phi_{+} = \alpha e^{-|k|z}$ (top)

$V_x = \frac{d\xi}{dt}$ $\phi_{-} = \beta e^{|k|z}$ (bottom)

$\xi = d e^{ik(x-ct)}$

take top: kinematic BC take $k > 0$

$z=0$

$$-\cancel{e^{ik(x-ct)}} \alpha |k| \cancel{e^{-|kz|}} = \partial_t \xi + (U \cdot \nabla)\xi$$

$$= -ikc \cancel{\xi} + U ik \xi$$

$$\boxed{\alpha = id(c-U)}$$

$$\boxed{\phi_{+} = i(c-u)\xi}$$

Likewise $\cancel{e^{ik(x-ct)}} \beta |k| e^{|kz|} = \partial_t \xi - (U \cdot \nabla)\xi$

$$\boxed{\beta = -id(c+u)}$$

$$\boxed{\phi_{-} = -i(c+u)\xi}$$

c) $\rho \frac{dV}{dt} = -\nabla P$

$$\partial_t V + (V \cdot \nabla)V + \frac{\nabla P}{\rho} = 0$$

Linearize

$$\partial_t \nabla\phi + U \cdot \nabla^2 \phi + \frac{\nabla P}{\rho} = 0$$

$$\nabla\left(\partial_t \phi + U \cdot \nabla \phi + \frac{P}{\rho}\right) = 0$$

$$\partial_t \phi + \vec{U}\cdot\nabla\phi + \frac{P}{\rho} = 0$$

$P_1 = P_2$ so

& $\rho_1 = \rho_2$ at boundary

$$\phi_+ = i\alpha(c-u)e^{-|k|z} e^{ik(x-ct)}$$

$$\phi_- = -i\alpha(c+u)e^{|k|z} e^{ik(x-ct)}$$

↑ eval @ 0

$$\frac{\phi_-}{\phi_+} = \frac{\beta}{\alpha} = \frac{-(c+u)}{c-u}$$

$$\partial_t \phi_+ + U\cdot\frac{\partial}{\partial x}\phi_+ = \partial_t\phi_- - U\cdot\frac{\partial}{\partial x}\phi_-$$

$$-ikc\,\phi_+ + U\,ik\,\phi_+ = -ikc\,\phi_- - U(ik)\phi_-$$

$$-(u-c)^2 = (c+u)^2$$

$$0 = (c+u)^2 + (u-c)^2$$

$$\boxed{c = \pm iu}$$

Purely growing (or damping)

d) Let $\vec{B} = (B,0,0)$ but $\delta\vec{B} = (B_x, B_y, B_z)$

$$U = (\pm U, 0, 0), \quad \delta U = (u_x, u_y, u_z)$$

Induction $\partial_t B = \nabla\times(U\times B) = (\vec{B}\cdot\vec{\nabla})\vec{u} - (\vec{U}\cdot\vec{\nabla})\vec{B}$

$$\partial_t B_x = B\frac{\partial u_x}{\partial x} \mp U\cdot\frac{\partial B_x}{\partial x} \qquad u_x = \nabla\phi$$

Assume $B_x = B_x e^{ik(x-ct)}$

$$-\cancel{ik}c\,B_x = B(ik)^2\phi \mp U\,\cancel{ik}\,B_x$$

$$-cB_x \pm uB_x = ik\,B\phi$$

$$\boxed{B_x = \frac{ik\,B\phi}{(-c\pm u)}}$$

e) Onto the Momentum balance $\left(\partial_t\phi + \vec{U}\cdot\nabla\phi + \frac{P}{\rho} = 0\right)$

We can do a litte trick to get the dynamic B.C. $+ \,\&\, - \frac{B_0 B_x}{\rho\mu}$

We get

$$\partial_t \phi_+ + U \frac{\partial \phi_+}{\partial x} - \frac{B B_{x+}}{\rho\mu} = -\frac{p}{\rho} - \frac{B B_x}{\rho\mu} = \partial_t \phi_- - U \frac{\partial \phi_-}{\partial x} - \frac{B B_{x+}}{\rho\mu}$$

$$-\cancel{ik}c\,\cancel{\phi_+} + U\,\cancel{ik}\,\cancel{\phi_+} - V_A^2 \frac{\cancel{ik}\,\cancel{\phi_+}}{-C+U} = -\cancel{ik}c\,\cancel{\phi_-} - U\,\cancel{ik}\,\cancel{\phi_-} - V_A^2 \frac{\cancel{ik}\,\cancel{\phi_-}}{-C-U}$$

$$-C+U-\frac{V_A^2}{-C+U} = \left(-(C+U) + \frac{V_A^2}{U+C}\right)\frac{-(C+U)}{C-U} \qquad \left(\frac{\phi_-}{\phi_+} = \frac{\beta}{\alpha} = \frac{-(C+U)}{C-U}\right)$$

$$(U-C)^2 - V_A^2 = -(C+U)^2 + V_A^2$$

$$(U-C)^2 + (C+U)^2 = 2V_A^2$$

$$U^2 - 2UC + C^2 + C^2 + 2UC + U^2 = 2V_A^2$$

$$2(U^2+C^2) = 2V_A^2$$

$$\boxed{C^2 = V_A^2 - U^2}$$

so $\boxed{U > V_A \text{ Unstable}}$ i.e. c is imaginary