II.4 Electrostatic Waves in Magnetized Plasma [50 pts]

Consider a homogeneous stationary hot (but nonrelativistic) plasma immersed in a homogeneous stationary magnetic field $\boldsymbol{B}_0$ directed along the z axis. In this problem, you are asked to study waves in this plasma under the electrostatic approximation.

(a) [5 pts] Explain what the electrostatic approximation is. Present a sufficient condition under which the electrostatic approximation is valid for a wave with given frequency ω and wave vector $\boldsymbol{k}$ in a medium with a given dielectric tensor ϵ.

(b) [15 pts] General electrostatic waves in magnetized plasma are governed by the so-called Harris dispersion relation $\mathscr{D}(\omega, \boldsymbol{k}) = 0$, where $\mathscr{D}(\omega, \boldsymbol{k}) = 1 + \sum_s X_s$ and

$$X_s = \sum_s \frac{\omega_{ps}^2}{k^2} \sum_{n=-\infty}^{\infty} \int d^3v \, \frac{J_n^2(k_\perp v_\perp/\Omega_s)}{\omega - k_\parallel v_\parallel - n\Omega_s} \left(\frac{n\Omega_s}{v_\perp} \frac{\partial f_{0s}}{\partial v_\perp} + k_\parallel \frac{\partial f_{0s}}{\partial v_\parallel} \right). \tag{2}$$

The standard notation is assumed here. In particular, s is the species index, $v_\perp = \sqrt{v_x^2 + v_y^2}$, and J_n are Bessel functions, which satisfy $J_n(x) = J_{-n}(x)$ for all x. **Assuming that Eq. (2) is given** and $f_{0s}(\boldsymbol{v}) = (2\pi v_\perp)^{-1}\delta(v_\parallel)\delta(v_\perp - V_s)$, show that

$$X_s = -\beta_s \sum_{n=0}^{\infty} \left\{ \frac{n^2}{q_s^2 - n^2} \frac{2[J_n^2(b_s)]'}{b_s} \sin^2\theta + \frac{q_s^2 + n^2}{(q_s^2 - n^2)^2} (2 - \delta_{n,0}) J_n^2(b_s) \cos^2\theta \right\}, \tag{3}$$

where $\beta_s = \omega_{ps}^2/\Omega_s^2$, $b_s = k_\perp V_s/\Omega_s$, $q_s = \omega/\Omega_s$, θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$, and $\delta_{n,0}$ is the Kronecker symbol (i.e., $\delta_{0,0} = 1$ and $\delta_{n\neq 0,0} = 0$).

(c) [15 pts] Derive the cold limit of Eq. (3) using that $J_n(x) \approx (x/2)^n/n!$ at small x. (See also Fig. 1 below.) Show that it leads to the same electrostatic dispersion relation as the one that flows from the standard cold-plasma dielectric tensor. Give an example of waves described by this dispersion relation. Show that in the cold limit, the group velocity and the phase velocity of a wave are mutually orthogonal. *Hint:* Recall how to express the group velocity from $\mathscr{D}(\omega, \boldsymbol{k}) = 0$ for general $\mathscr{D}$.

(d) [15 pts] Anisotropic distribution can be susceptible to the so-called Harris instability. Here, you are asked to derive this instability using your result from part (b). For simplicity, assume $\theta = \pi/2$, neglect the ion contribution, and keep only the leading-order corrections with respect to the electron gyroradius ρ_e. (You are expected to know what this means.) Calculate the maximum growth rate of the Harris instability and show that the corresponding wavelength is about $2.4\rho_e$. *Hint:* Consider using Fig. 1 to approximately evaluate the numerical coefficients.

a) Electro Static means $E = \nabla \phi$

A Sufficient Condition is that $N^2 \gg \epsilon_{ab}$ for all ab

or $E_{\parallel} \gg E_{\perp}$ so $\vec{E} \approx E_{\parallel}\hat{k} = i\vec{k}\phi = -\vec{\nabla}\phi$

b) $f_{0s}(\vec{v}) = \frac{1}{2\pi v_\perp}\delta(v_\parallel)\,\delta(v_\perp - V_s)$

$$d^3v = 2\pi V_\perp dV_\perp dV_\parallel$$

$\frac{k_\parallel^2}{k^2} = \cos^2\theta$ & $\frac{k_\perp^2}{k^2} = \sin^2\theta$

Start by IGBP. Boundary term $= 0$

$$X_s = \sum_s \frac{\omega_{ps}^2}{k^2} \sum_{n=-\infty}^{\infty} \int d^3v \, \frac{J_n^2(k_\perp v_\perp/\Omega_s)}{\omega - k_\parallel v_\parallel - n\Omega_s} \left(\frac{n\Omega_s}{v_\perp}\frac{\partial f_{0s}}{\partial v_\perp} + k_\parallel \frac{\partial f_{0s}}{\partial v_\parallel} \right).$$

$$X_s = -\frac{\omega_{ps}^2}{k^2}\sum_{-\infty}^{\infty}\int 2\pi V_\perp dV_\perp dV_\parallel \left[f_{0s}\frac{n\Omega_s}{\omega - k_\parallel V_\parallel - n\Omega_s}\frac{\partial}{\partial V_\perp}\left(J_n^2(\,)\right) + V_\perp f_{0s} k_\parallel J_n^2(\,)\frac{\partial}{\partial V_\parallel}\left(\frac{1}{\omega - k_\parallel V_\parallel - n\Omega}\right)\right]$$

$$\downarrow \frac{k_\parallel}{(\omega - k_\parallel V_\parallel - n\Omega)^2}$$

evaluate integral @ δ function locations

differentiation w.r.t. b
unstated info in problem

$$X_s = -\frac{\omega_{ps}^2}{k^2}\sum_{-\infty}^{\infty}\left(\frac{1}{V_s}\frac{n\Omega_s}{\omega - n\Omega_s}\left[J_n^2(b)\right]'\frac{k_\perp}{\Omega} + k_\parallel^2\frac{J_n^2(b)}{(\omega - n\Omega)^2}\right)$$

$$X_s = -\frac{\omega_{ps}^2}{k^2}\sum_{-\infty}^{\infty}\left(\frac{1}{V_s}\frac{n}{q-n}\left[J_n^2(b)\right]'\frac{k_\perp}{\Omega} + k_\parallel^2\frac{J_n^2(b)}{\Omega^2(q-n)^2}\right)$$

$$X_s = -\beta_s\sum_{-\infty}^{\infty}\left(\frac{1}{V_s}\frac{n\Omega^2}{(q-n)k^2}\left[J_n^2(b)\right]'\frac{k_\perp}{\Omega} + \cos^2\theta\frac{J_n^2(b)}{(q-n)^2}\right)$$

$$X_s = -\beta_s\left(\cos^2\theta\frac{J_0^2(b)}{q^2} + \sum_{n=1}^{\infty}\frac{[J_n^2(b)]'}{V_s}\frac{\Omega k_\perp}{k}\left(\frac{n}{q-n} - \frac{n}{q+n}\right) + \cos^2\theta J_n^2(b)\left(\frac{1}{(q-n)^2} + \frac{1}{(q+n)^2}\right)\right)$$

$$\frac{q^2+0^2}{(q^2-0^2)^2}$$

$$\frac{\Omega}{V_s k_\perp}\frac{k_\perp^2}{k^2}\left(\frac{\cancel{qn} + n^2 - \cancel{qn} + n^2}{(q^2-n^2)}\right)$$

$$\left(\frac{(q+n)^2 + (q-n)^2}{(q^2-n^2)^2}\right)$$

$$\frac{n^2}{(q^2-n^2)}\frac{2\left[J_n^2(b)\right]'}{b}\sin^2\theta + \frac{2(q^2+n^2)}{(q^2-n^2)^2}\cos^2\theta J_n(b)$$

$$X_s = -\beta_s \sum_{n=0}^{\infty} \left\{ \frac{n^2}{q_s^2 - n^2}\frac{2[J_n^2(b_s)]'}{b_s}\sin^2\theta + \frac{q_s^2+n^2}{(q_s^2-n^2)^2}(2-\delta_{n,0})J_n^2(b_s)\cos^2\theta \right\}, \quad (3)$$

c) $\chi_s = -\beta_s \sum_{n=0}^{\infty} \left\{ \frac{n^2\ 2}{q_s^2 - n^2} \left(\frac{[J_n^2(b_s)]'}{b_s} \right) \sin^2\theta + \frac{q_s^2 + n^2}{(q_s^2 - n^2)^2} (2 - \delta_{n,0}) J_n^2(b_s) \cos^2\theta \right\}$

From looking at the plots, we can see that $1 \approx J_0^2(b) >> J_{n \geq 1}^2(b)$ so J_0 term dominates. Also, $\frac{1}{2} \simeq [J_1^2(b)]'/b << [J_{n\geq 1}^2(b)]'/b$ so $[J_1^2(b)]'/b$ dominates

$$\chi_s = -\frac{\omega_{ps}^2}{\Omega_s^2} \left(\frac{1}{q_s^2 - 1} \sin^2\theta + \frac{1}{q_s^2} \cos^2\theta \right)$$

$\approx \frac{1}{q_s^2}$ large $\frac{\omega}{\Omega}$

$$\simeq -\frac{\omega_{ps}^2}{\Omega_s^2} \frac{\Omega_s^2}{\omega^2} (\sin^2\theta + \cos^2\theta) = -\frac{\omega_{ps}^2}{\Omega_s}$$

so $\epsilon = 1 - \frac{\omega_{ps}^2}{\omega^2}$ ✓ Cold Plasma ε

ex. Light wave or O mode

$$N^2 = 1 - \frac{\omega_{ps}^2}{\omega^2} = \frac{k^2 c^2}{\omega^2} \qquad \omega^2 = k^2 c^2 + \omega_{ps}^2$$

$$V_{ph} = \frac{\omega}{k} = \sqrt{c^2 + \omega_{ps}^2 / k^2} \qquad V_g = \frac{\partial \omega}{\partial k} = 2 \frac{2kc}{k^2 c^2 + \omega_{ps}^2}$$

$$\vec{k} \cdot \overline{\overline{\epsilon}}\, \vec{k} = 0 \qquad D_E\, E = 0$$

So $\left(1 - \frac{\omega_{ps}^2}{\omega^2}\right) k^2 = 0$

$$D = \frac{c^2}{\omega^2}(\hat{k}\hat{k} - I \hat{k}^2) + \varepsilon$$

$$= (\vec{N}_i \vec{N}_j - N^2 \delta_{ij}) + \hat{\varepsilon}_{ij}$$

Only diagonal term survives so $D = \varepsilon$

$$0 = \frac{\partial D}{\partial k} + \frac{\partial D}{\partial \omega} \cdot \frac{\partial \omega}{\partial k}$$

$\uparrow\ 0 = \frac{\partial \varepsilon}{\partial k}$ $\quad \uparrow\ V_g$ so $\frac{\partial D}{\partial \omega} \perp V_g$

$\uparrow$ $\epsilon = 1 + \frac{\omega_p^2}{\omega^2}$

$= -2 \frac{\omega_p^2}{\omega^3} \sim f(\omega)\, \hat{V}_{ph}$ direction So $V_g \perp V_{ph}$

$\uparrow$ some function

d) Not sure. For future generations