I.1: General Plasma Physics [25 points]

Consider a magneto-electric particle trap in the region $-L < z < L$. To accomplish this trap, suppose a magnetic field in the z direction such that

$$B = \begin{cases} B_0\left(1 + (R-1)(\frac{z}{L_m})^2\right), & \text{if } -L_m < z < 0; \\ B_0, & \text{if } z \geq 0. \end{cases}$$

Suppose also an electric potential

$$\phi = \begin{cases} 0, & \text{if } z < 0; \\ \phi_0\left(\frac{z}{L_e}\right)^2, & \text{if } 0 \leq z < L_e; \\ \phi_0, & \text{if } z > L_e. \end{cases}$$

(a) [**4 points**] Describe how ions might be trapped in this configuration of magnetic and electric fields. Would electrons also be trapped in the same fields?

(b) [**6 points**] Derive a trapping condition for confined particles in terms of the particle midplane perpendicular energy $W_{\perp 0}$ and midplane parallel energy $W_{\parallel 0}$, where these energies are defined at the axial location $z = 0$.

(c) [**2 points**] Sketch the trapping condition in $W_{\perp 0} - W_{\parallel 0}$ space.

(d) [**2 points**] If trapped ions of charge state q were scattered in pitch-angle, but not in energy, through collisions, from what end of the device would they leave? How does this answer depend on the midplane energy coordinates $W_{\perp 0}$ and $W_{\parallel 0}$? Please explain very briefly (in one sentence).

(e) [**5 points**] Suppose now that the electric potential is a varying function of time. Show that the second adiabatic invariant can be put in the form

$$W_{\parallel 0}^{1/2}\left(z_M + z_E\right) = \text{const.}$$

Here z_M and z_E are the turning points in the regions $z < 0$ and $z > 0$ respectively. What are z_M and z_E in terms of the parameters L_e, L_m, R, $W_{\perp 0}$, and $W_{\parallel 0}$. Define $W_c \equiv q\phi_0/(R-1)$. Show that, if $W_{\perp 0}/W_c \sim O(1)$, then $L_e \gg L_m$ implies $z_e \gg z_m$.

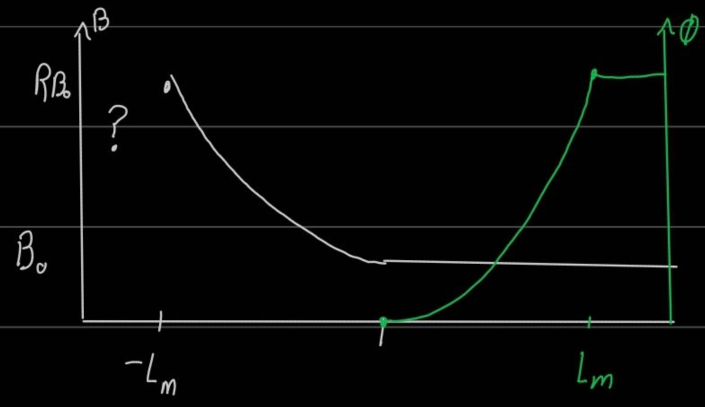
(f) [**6 points**] Suppose that the length $L_e(t)$ slowly changes in time, but assume that $L_e(t) \gg L_m$ for all t. Show that, if $L_e(t)$ is slowly shortened from $t = 0$ to $t = t_0$, such that $L_e(0)/L_e(t_0) = \alpha > 1$, then there is a region in $W_{\perp 0} - W_{\parallel 0}$ space (where coordinates are given at $t = 0$), such that any ions in that region will escape on a different side of the trap by the time $t = t_0$, than they otherwise would have eventually escaped by rare but finite pitch angle scattering had the trap potential not been altered ($\alpha = 1$). Show that this region is triangular in shape with area

$$A \simeq \frac{1}{2}(q\Phi_0)^2\left(1 - \alpha^{\overset{-1}{\cancel{1/2}}}\right)^2.$$

Nat knows this is a typo

Not so helpful hint: You may wish to use (but you do not really need it) the integral $\int_0^1((1-s^2)^{1/2} = \pi/4$.

a)



Ions are confined by $F = -\mu \nabla B$ force & reflected by $F = q\nabla\phi$ force. Electrons would not be reflected by the electrostatic potential, so they would not be confined

b) $$W_{\parallel 0} + W_{\perp 0} + q\overset{0}{\phi}(z=0) = W_{\parallel}(z) + W_{\perp}(z) + q\phi(z)$$

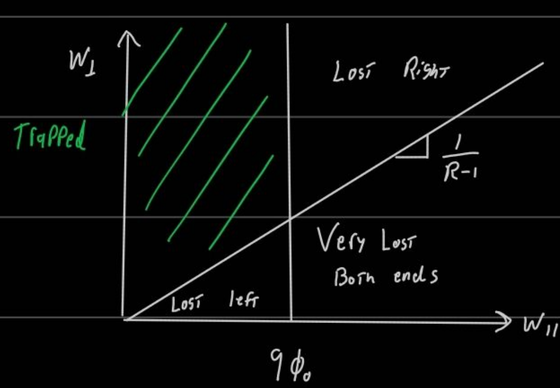
On Left $\quad W_{\parallel 0} + W_{\perp 0} < 0 + \mu B_{max} + 0 = R W_{\perp 0}$ trapped

So $W_{\parallel 0} < (R-1) W_{\perp 0}$ is trapped

On right $\quad W_{\parallel 0} + \cancel{W_{\perp 0}} < \overset{0}{W_{\parallel}(z_{max})} + \cancel{W_{\perp 0}} + q\phi_0$ trapped

$W_{\parallel 0} < q\phi_0$ are trapped

c)



d) It depends. Particles with $E_{tot} < q\phi_0$ would be lost on the magnetic side, but particles w/ $E_{tot} > q\phi_0$ would be lost on the electrostatic side

e) $\tilde{J} = \text{Const} = \oint V_\parallel \, dz$

$$W_\parallel + W_\perp + q\phi = \text{Const}$$

$$\frac{1}{2} m V_\parallel^2 + \mu B + q\phi = W_{\parallel 0} + W_{\perp 0}$$

$$V_\parallel = \sqrt{\frac{2}{m}\left(W_{\parallel 0} + \mu(B - B(z)) - q\phi(z)\right)}$$

$$\tilde{J} = \int_{z_m}^{0} dz \sqrt{\frac{2}{m}\left(W_{\parallel 0} + \mu\left(B_0 - B_0\left(1 + (R-1)\left(\frac{z}{L_m}\right)^2\right)\right)\right)} + \int_0^{z_E} dz \sqrt{\frac{2}{m}\left(W_{\parallel 0} - q\phi_0 \left(\frac{z}{L_e}\right)^2\right)}$$

Factor out. Const

$$\left(W_{\parallel 0} + W_{\perp 0} (1-R)\frac{z^2}{L_m^2}\right)^{1/2} \qquad + \sqrt{W_{\parallel 0}} \int_0^{z_E} dz \left(1 - \frac{q\phi_0}{W_{\parallel 0}} \left(\frac{z}{L_e}\right)^2\right)^{1/2}$$

$$\sqrt{\frac{2}{m} W_{\parallel 0}} \sqrt{\left(1 - \frac{W_{\perp 0}}{W_{\parallel 0}} (R-1) \frac{z^2}{L_m^2}\right)}$$

$$s = \sqrt{\frac{q\phi_0}{W_{\parallel 0}}} \frac{z}{L_e} \rightarrow ds = \sqrt{\frac{q\phi_0}{W_{\parallel 0}}} \frac{dz}{L_e} = \frac{dz}{z_E}$$

$$s(z_E) = 1$$

$$s(z_m) = \sqrt{\frac{W_{\perp 0}}{W_{\parallel 0}}(R-1)} \frac{1}{L_m}\left(L_m \sqrt{\frac{1}{R-1}\frac{W_{\parallel 0}}{W_{\perp 0}}}\right) = 1$$

Let $s = \sqrt{\frac{W_{\perp 0}}{W_{\parallel 0}}(R-1)} \frac{z}{L_m}$

$$ds = \sqrt{\frac{W_{\perp 0}}{W_{\parallel 0}}(R-1)} \frac{dz}{L_m} = \frac{dz}{z_M}$$

$$\sqrt{W_{\parallel 0}}\, z_E \underbrace{\int_0^1 (1-s^2)^{1/2}}_{\pi/4}$$

$$\sqrt{\frac{2}{m} W_{\parallel 0}} \int_1^0 (1-s^2)^{1/2} \sqrt{\frac{W_{\parallel 0}}{W_{\perp 0}} \frac{1}{R-1}} L_m \, ds$$

$$\sqrt{\frac{2}{m} \underbrace{\frac{W_{\parallel 0}}{W_{\perp 0}} \frac{1}{R-1}}_{z_m}} L_m \sqrt{W_{\parallel 0}} \frac{\pi}{4}$$

$$\sim z_m \sqrt{W_{\parallel 0}}$$

So $\sqrt{W_{\parallel 0}} \left(z_m + z_E \right) = \text{Const}$

Find z_m & z_E

$$0 = \sqrt{\frac{2}{m}\left(W_{\parallel 0} + \mu(B_0 - B(z_m)) - 0\right)} \qquad 0 = \sqrt{\frac{2}{m}\left(W_{\parallel 0} - q\phi(z_E)\right)}$$

$$W_{\parallel 0} = -\mu\left(B_0 - B_0\left(1 + (R-1)\left(\frac{z_m}{L_m}\right)^2\right)\right) \qquad 0 = W_{\parallel 0} - q\phi_0 \left(\frac{z_E}{L_e}\right)^2$$

$$\frac{W_{\parallel 0}}{\mu B_0} = (R-1)\left(\frac{z_m}{L_m}\right)^2 \qquad \frac{W_{\parallel 0}}{q\phi_0} L_m^2 = z_E^2$$

$$z_m^2 = L_m^2 \frac{1}{1-R} \frac{W_{\parallel 0}}{\mu B_0} = L_m^2 \frac{1}{R-1} \frac{W_{\parallel 0}}{W_{\perp 0}}$$

$$W_c = \frac{q\phi_0}{R-1} = \frac{W_{\parallel 0} L_E^2}{z_E^2} \frac{z_m^2}{L_m^2} \frac{W_{\perp 0}}{W_{\parallel 0}} = \frac{z_m^2}{z_E^2} W_{\perp 0} \frac{L_E}{L_m}$$

$$\frac{W_c}{W_{\perp 0}} = \frac{Z_m^2}{Z_E^2}\frac{L_E}{L_m}$$ $\uparrow \mathcal{O}(1)$ so

$$\frac{L_m}{L_E} = \frac{Z_m^2}{Z_E^2}$$

So $\frac{L_m}{L_E} \ll 1$ implies $\frac{Z_m}{Z_E} \ll 1$

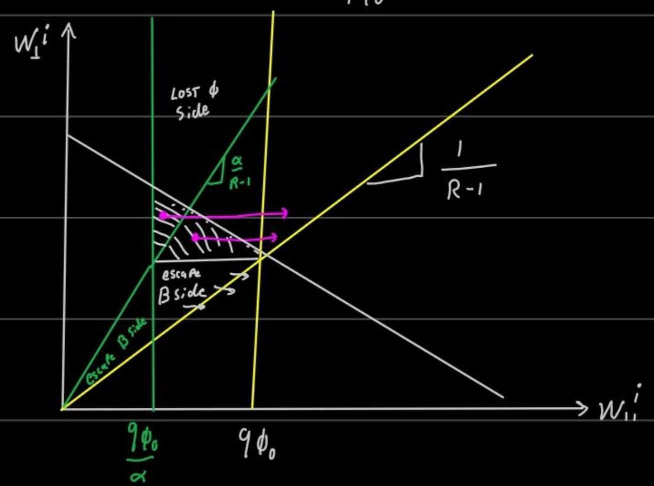
f) $W_{\parallel}^{1/2}(Z_m + Z_E) = \text{Const}$

Nat knows there is a typo in the Problem statement.

$$W_{\parallel}^{i\,1/2} Z_E^i = W_{\parallel}^{f\,1/2} Z_E^f \qquad \frac{W_{\parallel 0}}{q\phi_0} L_m^2 = Z_E^2$$

$$\sqrt{\frac{W_{\parallel}^i}{W_{\parallel}^f}} = \frac{Z_E^f}{Z_E^i} = \frac{L_E^f \sqrt{W_{\parallel}^f}}{L_E^i \sqrt{W_{\parallel}^i}} \qquad \frac{L_c^i}{L_c^f} = \alpha \qquad \frac{W_{\parallel 0}}{q\phi_0} L_E^2 = Z_E^2$$

So $W_{\parallel_0}^f = \alpha\, W_{\parallel_0}^i$



$W_{\parallel_0}^f < q\phi_0$ trapped

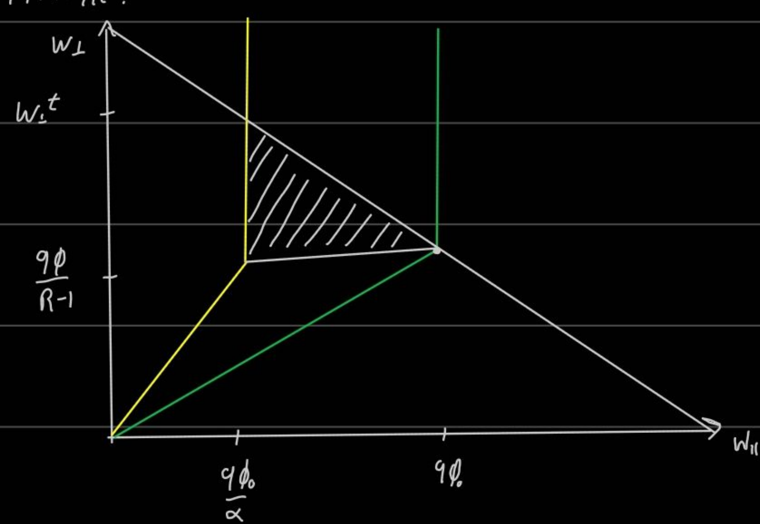
$\alpha W_{\parallel 0}^i < q\phi_0$ trapped

$W_{\parallel 0}^i < \frac{q\phi_0}{\alpha}$ trapped

$$W_{\parallel_0}^f \frac{1}{R-1} = W_{\perp 0}^f$$

$$W_{\parallel 0}^i \frac{\alpha}{R-1} = W_{\perp 0}^i$$

What is the top point of the Triangle?



$W_{\parallel} + W_\perp = \text{Const}$

$$q\phi_0 + \frac{q\phi_0}{R-1} = \frac{q\phi_0}{\alpha} + W_\perp^t$$

$$q\phi_0\left(1 + \frac{1}{R-1} - \frac{1}{\alpha}\right) = W_\perp^t$$

$$h = W_\perp^t - q\phi/_{R-1} = q\phi\left(1 - \frac{1}{\alpha}\right)$$

$$b = q\phi_0 - \frac{q\phi_0}{\alpha} = q\phi\left(1 - \frac{1}{\alpha}\right)$$

$$A = \frac{1}{2} bh = \frac{(q\phi_0)^2}{2}\left(1 - \frac{1}{\alpha}\right)^2$$