

II.2 Irreversible Processes [45 pts]

Consider the following 1D-1V linearized Vlasov equation governing the evolution of the distribution function $f(t, z, v) = f_0(v) + \delta f_k(t, v) \exp(ikz)$, where $f_0 = n \exp(-v^2/v_{\text{th}}^2)/\sqrt{\pi}$ is a stationary, homogeneous, Maxwellian background and δf_k is a small-amplitude perturbation of wavenumber k excited by the source term that appears on the right-hand side:

$$\left(\frac{\partial}{\partial t} + ikv \right) \delta f_k + ikv \frac{q\varphi_k}{T} f_0 = \frac{2v}{v_{\text{th}}^2} a_k(t) f_0. \quad (1)$$

The notation is standard: n is the number density, q is the charge, T is the temperature, and $v_{\text{th}} \doteq (2T/m)^{1/2}$ is the thermal speed for particles of mass m . The electrostatic potential φ_k satisfies

$$\frac{q\varphi_k}{T} = \frac{\alpha}{n} \int_{-\infty}^{\infty} dv \delta f_k, \quad (2)$$

where α is a constant. The acceleration $a_k(t)$ drives velocity fluctuations stochastically at wavenumber k ; it is described statistically by its two-time correlation function,

$$\langle a_k(t) a_k^*(t') \rangle = \varepsilon v_{\text{th}}^2 \delta(t - t') \quad \text{with} \quad \langle a_k(t) \rangle = 0, \quad (3)$$

where ε is a constant. Answer the following.

- (a) [14 points] Show that the steady-state fluctuation level satisfies

$$\left\langle \left| \frac{q\varphi_k(t)}{T} \right|^2 \right\rangle = \frac{\varepsilon}{|k|v_{\text{th}}} \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \left| \frac{Z'(\zeta)}{D_\alpha(\zeta)} \right|^2, \quad (4)$$

where $\zeta \doteq \omega/|k|v_{\text{th}}$, $Z(\zeta)$ is the plasma dispersion function, and

$$D_\alpha(\zeta) \doteq \frac{1}{\alpha} + 1 + \zeta Z(\zeta) = \frac{1}{\alpha} - \frac{1}{2} Z'(\zeta)$$

is the dielectric function.

- (b) [7 points] Solve $D_\alpha(\zeta) = 0$ in the limit $\alpha \gg 1$ to find the approximate dispersion relation for a rapidly oscillating, weakly damped mode:

$$\text{Re}(\zeta) \approx \pm \sqrt{\frac{\alpha}{2}}, \quad \text{Im}(\zeta) \doteq \frac{-\gamma}{|k|v_{\text{th}}} \approx -\sqrt{\pi} \left(\frac{\alpha}{2} \right)^2 \exp \left(-\frac{\alpha}{2} \right). \quad (5)$$

- (c) [10 points] With $\alpha = k^{-2}(4\pi e^2 n/T) \doteq (k\lambda_{De})^{-2}$, equation (5) describes long-wavelength ($k\lambda_{De} \ll 1$) Langmuir oscillations in an electron-ion plasma with unresponsive ions. Substitute (5) into (4) and evaluate the integral to show that the electrostatic energy associated with such Langmuir fluctuations satisfies

$$\left\langle \frac{|E_k|^2}{8\pi n} \right\rangle \approx \frac{T}{2} \frac{\varepsilon}{2\gamma}. \quad (6)$$

- (d) [7 points] For a stochastically driven, weakly damped, harmonic oscillator in thermal equilibrium, $\varepsilon = 2\gamma$. Use this to interpret (6) physically in the context of the steady-state electrostatic fluctuation level in a weakly coupled plasma in thermal equilibrium. Namely, comment in detail on how such a steady state is achieved, what $a_k(t)$ and ε represent physically in this case, and why the electrostatic energy is equal to $T/2$ at long wavelengths. Finally, how do you expect the steady-state spectrum to scale with k at small wavelengths ($k\lambda_{De} \gg 1$) where Langmuir fluctuations are strongly damped?
- (e) [7 points] Provide two ways that you would modify $a_k(t)$, ε , and/or α so that equations (1)–(3) provided a more accurate model for how a real plasma generates and interacts with a thermal bath of long-wavelength Langmuir fluctuations. Explain your answer.

Possibly useless information:

$$\lim_{\epsilon \rightarrow +0} \frac{1}{x - a \pm i\epsilon} = \text{PV} \left(\frac{1}{x - a} \right) \mp i\pi \delta(x - a), \quad \lim_{\epsilon \rightarrow +0} \frac{\epsilon}{(x - a)^2 + \epsilon^2} = \pi \delta(x - a)$$

$$f(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t), \quad f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega), \quad \delta(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t}$$

$$Z(\zeta) \doteq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x - \zeta}, \quad Z'(\zeta) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{x e^{-x^2}}{x - \zeta}, \quad Z'' + 2\zeta Z' + 2Z = 0$$

$$\zeta \ll 1 : Z(\zeta) \simeq i\sqrt{\pi} \exp(-\zeta^2) - 2\zeta \left(1 - \frac{2}{3}\zeta^2 - \dots \right)$$

$$|\zeta| \gg 1 : Z(\zeta) \simeq i\sqrt{\pi} \sigma \exp(-\zeta^2) - \frac{1}{\zeta} \left(1 + \frac{1}{2\zeta^2} + \dots \right); \quad \sigma = \begin{cases} 0 & \text{Im}(\zeta) > |\text{Re}(\zeta)|^{-1} \\ 1 & |\text{Im}(\zeta)| < |\text{Re}(\zeta)|^{-1} \\ 2 & \text{Im}(\zeta) < -|\text{Re}(\zeta)|^{-1} \end{cases}$$



$$a) (-i\omega + ikv) \delta f_k = \left(-i\kappa v \frac{q\phi}{T} + \frac{2v}{V_{th}^2} a(t) \right) f_0$$

$$\delta f = \left(\frac{i\kappa v \frac{q\phi}{T}}{-i(\omega - kv)} + \frac{2v}{V_{th}^2} a(t) \right) f_0$$

$$\frac{q\phi}{T} = \frac{\alpha}{n} \int_{-\infty}^{\infty} dv \left(\frac{i\kappa v \frac{q\phi}{T}}{-i(\omega - kv)} + \frac{2v}{V_{th}^2} a(t) \right) \frac{n}{\sqrt{\pi} V_{th}} e^{-v^2/V_{th}^2}$$

$$\frac{1}{2} \frac{q\phi}{T} = \int_{-\infty}^{\infty} dv \left(\frac{i\kappa v \frac{q\phi}{T}}{\kappa v - \omega} \frac{e^{-v^2/V_{th}^2}}{\sqrt{\pi} V_{th}} + \frac{2va}{(\omega - kv)} \frac{e^{-v^2/V_{th}^2}}{\sqrt{\pi} V_{th}^3} \right)$$

$$\underbrace{\frac{q\phi}{T} \left(\frac{1}{\alpha} - \frac{1}{2} Z' \left(\frac{\omega}{\kappa V_{th}} \right) \right)}_{D} = \frac{-i a(t)}{k V_{th}^2} Z'(\xi)$$

$$\frac{q\phi}{T} = \frac{-i}{k V_{th}^2} a(t) \frac{Z'(\xi)}{D(\xi)}$$

$$\langle \left| \frac{q\phi}{T} \right|^2 \rangle = \frac{1}{k^2 V_{th}^4} \xi V_{th}^2 \left| \frac{Z'(\xi)}{D(\xi)} \right|^2$$

$$= \int \frac{dw}{2\pi} \frac{dw'}{2\pi} e^{i(\omega - \omega')t} \frac{1}{\kappa^2 V_{th}^4} \left| \frac{Z'(\xi)}{D(\xi)} \right|^2 \langle a_\omega a_{\omega'}^\dagger \rangle$$

$$dw = k V_{th} d\xi$$

$$\int \frac{dw}{2\pi} e^{-i\omega t} \frac{q\phi_{kv}}{T} \int \frac{dw'}{2\pi} e^{i\omega' t} \frac{q\phi}{T}$$

$$= \int dt dt' e^{i\omega t - i\omega t'} \langle a a^\dagger \rangle = \int \frac{dt}{2\pi} \frac{dt'}{2\pi} e^{i\omega t - i\omega t'} e^{V_{th}^2} \delta(t - t')$$

$$= 2\pi \delta(\omega - \omega') \xi V_{th}^2$$

$$= \int \frac{dw}{2\pi} \frac{dw}{2\pi} e^{i(w-w')t} \underbrace{\frac{1}{k^2 V_{th}}} \left| \frac{Z'(\xi)}{D(\xi)} \right|^2 \cancel{\delta(w-w')} \delta(w-w') \cancel{V_{th}}$$

$dw = k V_{th} d\xi$

$$= \frac{\epsilon}{k V_{th}} \int_0^\infty \frac{d\xi}{2\pi} \left| \frac{Z'(\xi)}{D(\xi)} \right|^2$$

b) $D = \frac{1}{\alpha} - \frac{1}{2} Z'(\xi)$ $\alpha \gg 1$ $\xi \gg 1$

$$Z'(\xi) \approx -2i\sqrt{\pi} \xi e^{-\xi^2} + \frac{1}{2\xi^2}$$

$$\frac{1}{\alpha} = -i\sqrt{\pi} \xi e^{-\xi^2} + \frac{1}{2\xi^2}$$

\uparrow
tiny

$$\frac{1}{\alpha} = \frac{1}{2\xi_{Re}^2}$$

$$\xi \sim \pm \sqrt{\frac{\alpha}{2}}$$

Calculate real root Then expand around the real root

$$\operatorname{Im}(D_\alpha(\xi_r)) = \sqrt{\pi} \sqrt{\frac{\alpha}{2}} e^{-\alpha/2}$$

$$\partial_\xi \operatorname{Re}(D_\alpha(\xi_r)) = \frac{1}{\xi_r^3} = \left(\frac{\xi}{\alpha}\right)^{3/2}$$

$$\xi_i = -\frac{\operatorname{Im}(D_\alpha(\xi_r))}{\partial_\xi \operatorname{Re}(D_\alpha(\xi_r))} = -\sqrt{\pi} \left(\frac{\alpha}{2}\right)^2 e^{-\alpha/2}$$

c) $\langle \left| \frac{q\phi}{T} \right|^2 \rangle = \frac{\epsilon}{k V_{th}} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \left| \frac{Z'(\xi)}{D_\alpha(\xi)} \right|^2$

$$Z' \approx \frac{1}{\xi^2}$$

$$D_\alpha \sim \frac{1}{\alpha} - \frac{1}{2\xi^2} + i\sqrt{\pi} e^{-\xi^2} \xi$$

$$\begin{aligned}
 I_{\text{Hegel}} &= \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} \left| \frac{\alpha}{\xi^2 - \frac{\alpha^2}{4} + i\pi e^{-\xi^2/4}} \right|^2 \\
 &= \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{\alpha^2}{\left(\xi^2 - \frac{\alpha^2}{4}\right)^2 + \pi^2 e^{-2\xi^2}} \\
 &\quad \uparrow \qquad \qquad \qquad \text{Simplify} \\
 &\quad \text{dominant at the pole} \\
 &\quad \xi = \pm \sqrt{\frac{\alpha}{2}}
 \end{aligned}$$

It's like 2 delta functions

$$\text{Near the pole} \quad \frac{\alpha}{\left(\xi_0 - \sqrt{\frac{\alpha}{2}}\right)^2 \left(\xi_0 + \sqrt{\frac{\alpha}{2}}\right)^2 + \pi^2 e^{-\xi_0^2} \alpha^2 \xi_0^6}$$

$$\frac{\alpha}{\left(\xi_0 + \sqrt{\frac{\alpha}{2}}\right)^2} \quad \frac{1}{\left(\xi - \sqrt{\frac{\alpha}{2}}\right)^2 + \pi^2 e^{-\xi_0^2} \alpha^2 \xi_0^6} \quad \frac{1}{\left(\xi_0 + \sqrt{\frac{\alpha}{2}}\right)^2}$$

$$\begin{aligned}
 &= 2 \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{\alpha^2}{\left(\xi - \sqrt{\frac{\alpha}{2}}\right)^2 + \pi^2 e^{-\xi_0^2} \alpha^2 \left(\frac{\alpha}{2}\right)^3} \\
 &\quad \underbrace{\frac{1}{\sqrt{\pi} \alpha \left(\frac{\alpha^3}{2}\right) e^{-\alpha/2}}}_{E} e^{-\alpha/2} \\
 &\quad \underbrace{\frac{\alpha}{\sqrt{\pi} \alpha \left(\frac{\alpha^3}{2}\right) e^{-\alpha/2}}_{\alpha^2} \left(\xi - \sqrt{\frac{\alpha}{2}}\right)^2 + \pi^2 e^{-\xi_0^2} \left(\frac{\alpha}{2}\right)^3 e^{-\alpha}}_{\alpha^2}
 \end{aligned}$$

$$\pi \delta(\zeta - \sqrt{\frac{\alpha}{2}})$$

$$= \frac{1}{\sqrt{\pi}} \left(\frac{\alpha}{2} \right)^{-1} e^{-\alpha/2}$$

$$\begin{aligned} \langle \frac{|E_k|^2}{8\pi n} \rangle &= \frac{1}{8\pi n} \frac{T^2 k^2}{q^2} \langle \left| \frac{q\phi}{T} \right|^2 \rangle = \underbrace{\frac{1}{8\pi n} \frac{T^2 k^2}{q^2}}_{\frac{T}{2\alpha}} \boxed{\sqrt{\pi} e^{-\alpha/2} \left(\frac{\alpha}{2} \right)^3} \frac{\epsilon}{k V_{th}} \\ &= \frac{T}{2\alpha} \frac{\epsilon}{2} \end{aligned}$$

$$d) \quad \epsilon = 2 T \quad \text{so} \quad \langle \frac{E^2}{8\pi n} \rangle = \frac{T}{2}$$

- Discrete particle excited, then Landau damped into δf
The kick \wedge
Excites plasma wave

Energy dissipated into thermal energy, which excites particles, & shear

- Fluctuation Dissipation Theorem

What is $a(k)$ $\delta a \cdot \partial r f$ discrete particle

fluctuating electric field white noise

Stochastic Langmuir waves

ϵ - interaction energy. Fluctuation energy

$$W = \int d\mathbf{k} \frac{T}{2} \frac{1}{1 + k^2 \lambda_D^2} \sim \frac{1}{k^2}$$

- d) Velocity dependence of a_k for polarization drag

Finite correlation time, ω_p^{-1}

\propto only e^- , so odd ions

$\epsilon \rightarrow \epsilon_{kw}$ spectrum of interaction energy