Easy heuristic derivations for neoclassical quantities

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1 Trapping condition

Setting the mass of the particle to 1, we can state the trapping condition as

$$\frac{1}{2}\left.v_\parallel^2\right|_{\theta=0} + \frac{1}{2}\left.v_\perp^2\right|_{\theta=0} = \mu B_{ref} = \left.\left(\frac{1/2v_\perp^2}{B}\right|_{\theta=0}\right) B_{ref}. \tag{1}$$

Recall $B = B_0\left(1 - \epsilon\cos\theta\right)$, with $\epsilon = r/R$. The trapped-passing boundary is found by setting $B_{ref} = B_{max} = B(\theta = \pi) = B_0\left(1+\epsilon\right)$. So

$$\left.\frac{v_\parallel^2}{v_\perp^2}\right|_{\theta=0} = \frac{B_{max}}{B_{\theta=0}} - 1 = \frac{1+\epsilon}{1-\epsilon} - 1 \approx \epsilon, \tag{2}$$

or

$$\left.\left|\frac{v_\parallel}{v_\perp}\right|\right|_{\theta=0} \approx \epsilon^{1/2}. \tag{3}$$

2 Banana Orbit Width

The first method to find the banana orbit width utilizes the conservation of angular momentum. First note that the function $\psi = -RA_\phi$ is proportional to the poloidal flux function. From Diagnostics we found that $\nabla\psi = RB_P\hat{\boldsymbol{\psi}}$, where $\mathbf{B}_P = \nabla \times \mathbf{A}_\phi$. The conserved momentum is

$$P_\phi = mRv_\phi + eRA_\phi = mRv_\phi - e\psi. \tag{4}$$

This form is written in cylindrical coordinates. The calculations below will be done in toroidal coordinates, where $R = R_0 + r\cos\theta$. At the turning point $(r, \theta) = (r, \theta_t)$, we have

$$P_\phi = -e\psi(r). \tag{5}$$

At the midplane $(r, \theta) = (r + \Lambda, 0)$, we have

$$P_\phi = m(R_0 + r + \Lambda)v_\phi - e\psi(r + \Lambda). \tag{6}$$

But note that

$$\psi(r+\Lambda) \approx \psi(r) + \Lambda\hat{\mathbf{r}}\cdot\nabla\psi = \psi(r) + \Lambda(R_0+r)B_P. \tag{7}$$

Setting Eqs. 5 and 6 equal to one another and utilizing Eq. 7, we find:

$$\cancel{-e\psi(r)} \approx m(R_0+r+\Lambda)v_\phi - \cancel{e\psi(r)} - e\Lambda(R_0+r)B_P.$$

Thus we find

$$\Lambda \approx \frac{mv_\phi}{eB_P}\frac{R_0+r+\Lambda}{R_0+r} \approx \frac{mv_\phi}{eB_P}. \tag{8}$$

Now recall that

$$q = \epsilon\frac{B_T}{B_P},$$

and $|v_\phi| \approx \left|v_\parallel\right| \approx |v_\perp|\,\epsilon^{1/2} \approx v_{th}\epsilon^{1/2}$, so Eq. 8 becomes

$$\Lambda \approx \frac{mv_{th}\epsilon^{1/2}q}{e\epsilon B_T} = \rho q\epsilon^{-1/2}, \tag{9}$$

where ρ is the thermal gyroradius of the corresponding species.

The other method for calculating Λ requires first calculating the bounce frequency. Note that for a trapped particle, the bounce period is given by:

$$\tau_b = \frac{2\pi}{\omega_b} = 2\int_{-\theta_0}^{\theta_0}\frac{ds}{\left|v_\parallel(\theta)\right|} = 2\int_{-\theta_0}^{\theta_0}\frac{R_0 q d\theta}{\sqrt{2E\left(1-\lambda(1-\epsilon\cos\theta)\right)}}, \tag{10}$$

where $ds = R_0 q d\theta$ is the differential distance traversed along a field line over the interval $d\theta$, E is the particle energy, $\lambda = \mu B_0/E$, and $\left|v_\parallel(\theta)\right|$ is calculated according to:

$$\frac{1}{2}v_\parallel^2 = E - \frac{1}{2}v_\perp^2 = E - \mu B_0(1-\epsilon\cos\theta).$$

We can once again use our approximation $\left|v_\parallel\right| \approx v_{th}\epsilon^{1/2}$, and finally we have

$$\tau_b \sim \frac{R_0 q}{v_{th}\epsilon^{1/2}},$$

or

$$\omega_b \sim \frac{v_{th}\epsilon^{1/2}}{R_0 q}. \tag{11}$$

Additionally, we need to know the approximate radial grad-B drift velocity, which is given by:

$$(v_d)_r = \frac{\mu}{\Omega_c}\left(\hat{\mathbf{n}}\times\nabla B\right)_r \sim \frac{v_{th}^2}{\Omega_c B}\frac{B}{R_0} \sim \frac{\rho v_{th}}{R_0}. \tag{12}$$

Thus, we can find how Λ scales by combining Eqs. 11 and 12:

$$\Lambda \sim \frac{(v_d)_r}{\omega_b} \sim \frac{\rho v_{th}}{R_0}\frac{R_0 q}{v_{th}\epsilon^{1/2}} \sim \rho q\epsilon^{-1/2}. \tag{13}$$

3 Pfirsch-Schluter (i.e. plateau regime) Diffusion

Classical diffusion in a toroidal device increases substantially even in the short mean-free-path limit. In this case we say that:

$$D_{P-S} \sim \frac{(\Delta x_{P-S})^2}{\Delta t_\parallel} = ((v_d)_r)^2 \Delta t_\parallel, \tag{14}$$

where $\Delta x_{P-S} = (v_d)_r \Delta t_\parallel$, and $\Delta t_\parallel$ is the parallel diffusion time. We already calculated $(v_d)_r$ in Eq. 12 above, so all that remains is the calculation of $\Delta t_\parallel$. Now, note that parallel diffusion scales like:

$$D_\parallel \sim \frac{(\Delta x_\parallel)^2}{\Delta t_\parallel} \sim \lambda_{mfp}^2 \nu_{ei}^{90^\circ} \sim \left(\frac{v_{th}}{\nu_{ei}^{90^\circ}}\right)^2 \nu_{ei}^{90^\circ} = \frac{v_{th}^2}{\nu_{ei}^{90^\circ}}. \tag{15}$$

If we say that $\Delta x_\parallel \sim R_0 q$, i.e. that it scales like the length of a field line around the torus, then we determine that

$$\Delta t_\parallel \sim \frac{\nu_{ei}^{90^\circ} (R_0 q)^2}{v_{th}^2}, \tag{16}$$

Combining the results of Eqs. 12 and 16, we find that

$$D_{P-S} \sim \left(\frac{\rho v_{th}}{R_0}\right)^2 \frac{\nu_{ei}^{90^\circ} (R_0 q)^2}{v_{th}^2} \sim \nu_{ei}^{90^\circ} \rho^2 q^2. \tag{17}$$

4 Banana Diffusion

The determination of the banana diffusion coefficient is simple. First, we note that in the strongly-trapped regime the effective collision frequency ν_{eff} goes like $\nu_{ei}^{90^\circ}/\epsilon$. Additionally, the flux-surface-averaged trapped-particle fraction $\langle f_T \rangle \sim \epsilon^{1/2}$, owing to our determination that the slope of the trapped-passing boundary in $v_\perp$-$v_\parallel$ space scales like $\epsilon^{1/2}$. If we look at a fixed-energy shell in three-dimensional v-space, with two perpendicular directions and one parallel direction, and assume we have an isotropic distribution, we find that of the 4π of solid angle subtending the shell, a fraction scaling like $\epsilon^{1/2}$ subtends the portion of the shell corresponding to trapped particles.

With this information, we find the simple result that

$$D_{ban} \sim \langle f_T \rangle \Lambda^2 \nu_{eff} \sim \nu_{ei}^{90^\circ} \rho^2 q^2 \epsilon^{-3/2}. \tag{18}$$

5 Ware Pinch

The presence of a parallel electric field $E_\parallel$ shifts the turning points of a banana orbit by an angle $\Delta\theta \approx \Delta v_\parallel / v_\parallel$, where we continue to approximate $v_\parallel \approx v_{th} \epsilon^{1/2}$.

$\Delta v_\parallel$ is approximately given by

$$\Delta v_\parallel \sim \frac{eE_\parallel}{m\omega_b} \sim \frac{eE_\parallel}{m} \frac{R_0 q}{v_{th}\epsilon^{1/2}}. \tag{19}$$

An effective net radial Ware velocity v_W can be approximated as:

$$v_W \sim (v_D)_r \, \Delta\theta \sim \left(\frac{v_{th}\rho}{R_0}\right) \left(\frac{eE_\parallel R_0 q}{m v_{th}\epsilon^{1/2}}\right) \left(\frac{1}{v_{th}\epsilon^{1/2}}\right) \sim \frac{cE_\parallel}{B_P}. \tag{20}$$

This effective velocity accounts for the asymmetric radial drift near the turning points and distributes the extra drift over the entire bounce interval, so that representing the net effect over many bounce intervals yields the same result in both the discrete and smoothed perspectives. The corresponding particle flux is given by

$$(\Gamma_W)_r = -\langle f_T \rangle n_0 v_W \sim -\frac{cE_\parallel}{B_P}\epsilon^{1/2} n_0. \tag{21}$$

6 Bootstrap Current

This heuristic derivation is somewhat different and more "back of the envelope" than Martin's, but one important difference is that it doesn't require setting $\nu_{ei} = \nu_{ee}$. We claim that the bootstrap current j_b is given by

$$j_b = ev_\parallel f_{c-c}, \tag{22}$$

where f_{c-c} is the difference in density between co-moving and counter-moving **passing** particles. Essentially, the trapped particles do not carry a significant net current because particles on a banana orbit bounce back and forth at the corresponding bounce frequency ω_b; time-averaging the effective current over an entire bounce interval for any particle will yield a net current of zero. However, the trapped particles establish a collisional equilibrium with co- and counter-moving passing particles, which do not suffer the same current-negating time-averaging fate as the trapped particles, such that the densities of co- and counter-moving passing particles are proportional to the densities of co- and counter-moving trapped particles. We estimate

$$f_{c-c} \sim \Lambda \frac{dn}{dr} \sim \rho q \epsilon^{-1/2} \frac{dn}{dr}. \tag{23}$$

In other words, the density difference between co- and counter-moving particles scales like the variation of the equilibrium density over a banana width. Finally, we get

$$j_b \sim ev_{th} \left(\frac{v_{th}}{\Omega_c} \cdot \epsilon \frac{B_T}{B_P} \cdot \epsilon^{-1/2} \frac{dn}{dr}\right) \sim \frac{\epsilon^{1/2} c}{B_P} T \frac{dn}{dr}. \tag{24}$$

Note that we used $v_\parallel \sim v_{th}$ here and not $v_\parallel \sim v_{th}\epsilon^{1/2}$ because the current is carried by the passing particles, not the trapped particles. The overwhelming majority of the passing particles have $v_\parallel$ much larger than $v_\parallel$ for the trapped particles.