What is the Bogoliubov Hierarchy?

What is $b_0$?

How can you determine whether a given collision operator conserves entropy?
The Bogoliubov Hierarchy is a hierarchy of time scales describing the relaxation of an arbitrary perturbation. A perturbed system will relax towards thermal equilibrium in 3 principal stages:

1. Pair correlations (due to shielding over distances $\lambda_0$) on a scale $T_{\lambda_0} \sim \frac{1}{\nu}$.

2. The velocity distribution relaxes to a local Maxwellian on the collisional time scale $T_c = \frac{1}{\nu}$ smoothing out on scales of order $\lambda_{mfp}$.

3. Hydrodynamic (diffusion) processes occur on macroscopic space and time scales $L \gg \lambda_{mfp}$, $t \gg T_c$ trying to relax the system to a global, space and time independent Maxwellian.

$$b_0 = \frac{e^2}{T}$$ is the distance of closest approach of 2 particles directly at each other. All kinetic energy has become potential energy:

$$\frac{1}{2} m v^2 = \frac{e^2}{b_0} \left( v \cdot v = \sqrt{T} \right)$$

$$b_0 = \frac{e^2}{T} \quad \text{(ignore factor of 2)}$$

Entropy:

$$S = \int f \ln f \, df$$

$$\therefore \left( \frac{\partial S}{\partial t} \right)_{\text{coll}} = \int \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \left( \ln f + 1 \right) \, df$$

$$= -\int \left( C f \right) \left( \ln f + 1 \right) \, df$$
Give a heuristic argument for the relationship between $v_i e$ and $v_i$.

How can you estimate the friction force, $\dot{R}$, and what do you get?

How can you show that a collision operator, $C_s$, conserves momentum?
From momentum conservation,

\[ m_e n_e \left( \frac{\partial \mathbf{v}_e}{\partial t} \right)_{\text{coll}} = -m_e n_e \left( \frac{\partial \mathbf{u}_i}{\partial t} \right)_{\text{coll}} \]

And, by definition:

\[ \left( \frac{\partial \mathbf{v}_e}{\partial t} \right)_{\text{coll}} = -\mathbf{v}_e \cdot (\mathbf{u}_e - \mathbf{U}_i) \]

\[ \left( \frac{\partial \mathbf{u}_i}{\partial t} \right)_{\text{coll}} = -\mathbf{v}_i \cdot (\mathbf{U}_i - \mathbf{u}_e) \]

So:

\[ \mathbf{v}_e = \left( \frac{m_e n_e}{m_i n_i} \right) \mathbf{v}_i \]

\[ \overline{R} = -\int d\mathbf{v}_e \, (m \bar{n}) e \, \mathbf{v}_e \, C_{ei} \, [f] \]

With the Lorentz operator for \( C_{ei} \), you can estimate \( \overline{R} \) by taking \( f_e \) to be a shifted Maxwellian (a 0-centered Maxwellian would have zero friction force). Also assume that \( \mathbf{v}_e \ll \mathbf{U}_i \).

You get:

\[ \overline{R} \approx -m_e n_e \mathbf{v}_i \cdot (\mathbf{u}_e - \mathbf{U}_i) \]

to show that a collision operator, \( C_{ei} [f] \), conserves momentum, you must show that:

\[ \left( \frac{\partial \mathbf{p}}{\partial t} \right)_{\text{coll}} = \sum_s \int d\mathbf{v} \, (m \bar{n})_s \, \mathbf{v} \cdot C_{s} [f] = 0 \]
How do you show that a collision operator $C_s[f]$ conserves kinetic energy?

How do you show that a collision operator, $C_s[f]$ conserves number of particles?

Derive a heuristic scaling for $v_{ee}$, the electron-ion collision frequency.
to show that a collision operator $C_s \left[ f \right]$ conserves kinetic energy, you must show:

$$\left( \frac{dE}{dt} \right)_{coll} = - \sum s \int d\mathbf{\nabla} \frac{1}{2} (\mathbf{n} \cdot \mathbf{v}) v' C_s \left[ f \right] = 0$$

To show that a collision operator $C_s \left[ f \right]$ conserves number of particles, you must show that:

$$\left( \frac{dn}{dt} \right)_{coll} = - \overline{n}_s \int d\mathbf{\nabla} C_s \left[ f \right] = 0$$

use $v \sim n \sigma v$,

take $n \sim n_i$, consider ions nearly immobile, $v \sim v_{Te}$, $\sigma \sim b_o^2$ where $b_o$ is the distance at which potential energy of the interaction equals electron thermal energy, $b_o \sim \frac{2e^2}{m_e}$. Also, don't forget the factor of $\frac{2}{\ln \Lambda}$ due to dominance of small 4- wave scattering:

$$v_{Te} \sim n_i b_o v_{Te} \ln \Lambda \sim \frac{n_i e^2 e^2}{m_e} \ln \Lambda$$
Why is it kinetic energy, not total energy, that the Landau (Lorentz, Balescu-Leonard, etc.) collision operator conserves?

How would this inner product be defined in the Chapman-Enskog procedure:

\[ \langle \Psi | \hat{L} | \chi \rangle = ? \]

What is this:

\[ C_{\Psi\chi} = -V(v) \frac{\partial}{\partial x} (1-x^2) \frac{\partial f}{\partial x} \]

- What is \( x \)?
- What is \( V(v) \)?
- Where did it come from?
- What does it describe?
because we assumed straight line trajectories, with constant $v$.

To first order, there was no change in potential energy.

for an entire plasma fluid, in 3-D:

$$\langle \hat{C}_1 | \hat{X} \rangle \equiv \sum_3 \int d^3v \, \psi(\mathbf{v}) \hat{C}_1 \times (\mathbf{v}) \, f_0(\mathbf{v})$$

$\hat{C}_1$ acts on $(X \, f_0)$

for a test particle, in 1-D:

$$\langle \hat{C}_1 | X \rangle \equiv \int dv \, \psi(\mathbf{v}) \hat{C}_1 \times X \, f_0$$

**Lorentz collision operator:** It came from the Landau collision operator, with the assumption that $m_e \ll m_i$.

$$\lambda = \frac{v_e}{v} \cos \theta$$

is the pitch angle parameter.

$\mathcal{D}(v) = \mathcal{D}(v_e) (\frac{v_e}{v})^3$ where $\mathcal{D}(v_e)$ is as derived with a heuristic scaling argument.

*Note:* $v^3$ dependence can lead to runaway electrons. The Lorentz operator conserves electron kinetic energy.

It describes the scattering of electrons off of immobile ions - occurs at the constant electron energy "on the energy shell"
How could you find all conservation laws of a given collision operator?

What does the Landau operator conserve?

What is the definition of conductivity (Tensor and Scalar)?
to find all quantities which are conserved by
a given collision operator, write:

\[ \int d^3v \ W(\vec{v}) \ C[f] = 0 \]

and try to solve for \( W \) for arbitrary \( f \).

→ Try to get in form: \( \int d^3v \ G(W) \cdot f = 0 \). Then \( G(W) = 0 \)

* integrate by parts to remove all derivatives from \( f \).

The Landau operator conserves density
momentum and kinetic energy and nothing else.

\text{conductivity tensor is defined by:} \[ \vec{J}_{\vec{k},\omega} = \vec{D}_{\vec{k},\omega} \cdot \vec{E}_{\vec{k},\omega} \]

with no asymmetry, this reduces to:

\[ \vec{J}_{\vec{k},\omega} = \sigma(\vec{k},\omega) \vec{E}_{\vec{k},\omega} \] where \( \sigma = \hat{k} \cdot \hat{\sigma} \cdot \hat{k} \)

and the \( \vec{E} \) field is given by:

\[ \vec{E}(x,t) = \vec{E}_{\vec{k},\omega} \exp(i \vec{k} \cdot \vec{x} - \omega t) \]
How are $\langle v^2 \rangle$ and $T$ related in thermal equilibrium?

What does

$$f_k(\hat{r}, \vec{v}, t) \, d\hat{r} \, d\vec{v}$$

represent?

In the collisionless limit, does the conductivity have a dissipative part?
\[ \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} T \text{ in thermal equilibrium} \]

\[ f_\alpha(\mathbf{r}, \mathbf{v}, t) \, d\mathbf{r} \, d\mathbf{v} \text{ represents the number of particles of species } \alpha \text{ in a 6-D volume element } d\mathbf{r} \, d\mathbf{v} \]

Yes; it is due to Landau damping.

(can use Vlasov dielectric in the collisionless limit)
What are runaway particles?

Why is it possible to have different electron and ion temperatures in one plasma?

How is the dielectric function defined, in terms of total and test fields/potentials?
Runaway particles are particles that accelerate indefinitely and never reach a steady state. This can arise because the Coulomb scattering cross section decreases rapidly at high energies: $\sigma \sim \left( \frac{V_{th}}{v} \right)^3$.

It is possible to have different electron and ion temperatures in one plasma because the large mass difference ensures that each particle species will thermalize with itself much sooner than the two will thermalize together.

$E_{\vec{k},\omega}^{(\text{ion})} = \frac{E_{\vec{k},\omega}^{(\text{test})}}{\mathcal{D}(\vec{k},\omega)}$

or, by Poisson's law:

$\delta \Phi_{\vec{k},\omega}^{(\text{ion})} = \frac{\delta \Phi_{\vec{k},\omega}^{(\text{test})}}{\mathcal{D}(\vec{k},\omega)}$
What does the test particle superposition principle state, and what does it allow you to compute?

What is the charge density of a test particle:
- in configuration space?
- in Fourier space?

What is the total charge density of a single quasiparticle?
The test particle superposition principle states that, to lowest order in the plasma parameters, one may consider the plasma to be composed of a collection of quasiparticles; statistically independent shielded test particles. Correlation effects accounted for by shielding.

It can be used to compute two-point quantities (e.g., fluctuation spectrum of the electric field) correctly through $O(\varepsilon_p)$.

\[
\rho^{\text{test}}(\hat{x}, t) = Q \delta(\hat{x} - \hat{x}_0 - \hat{v}t)
\]

\[
\rho^{\text{test}}(k, \omega) = \int d\hat{x} \int dt \ e^{-i(\hat{k} \cdot \hat{x} - \omega t)} Q \delta(\hat{x} - \hat{x}_0 - \hat{v}t)
\]

\[
= 2\pi Q e^{-ik \cdot \hat{x}_0} \delta(\omega - \hat{k} \cdot \hat{v})
\]

\[
\rho^{Qp}(k, \omega) = \frac{\rho^{\text{test}}(k, \omega)}{\mathcal{D}(k, \omega)}
\]

Total is sum of induced + test.
What is the electric field of a quasiparticle?

How do you use the assumption of statistical independence of quasiparticles to simplify:

\[ \langle E^{tot}(R,\epsilon)E^{tot}(-R,\epsilon) \rangle = \sum_i \sum_j \langle E_i^{op}(R,\epsilon)E_j^{op}(-R,\epsilon) \rangle \]

If \( A_i \) and \( A_j \) are statistically independent, how can you express:

\[ \sum_i \sum_j A_i A_j = ? \]
\[ i \vec{k} \vec{E}^{\text{op}}(\vec{r}, \omega) = 4\pi \rho^{\text{op}}(\vec{r}, \omega) \]

or, defining \( \vec{E}_r = \frac{4\pi i \vec{k}}{\vec{k}^2} \)

\[ \vec{E}^{\text{op}}(\vec{r}, \omega) = \vec{E}_r \rho^{\text{op}}(\vec{r}, \omega) \]

\[
\langle \vec{E}(\vec{r}, \omega) \cdot \vec{E}(\vec{r}', \omega') \rangle = \sum_c \sum_{\vec{k}} \langle E_c^{\text{op}}(\vec{k}, \omega) E_{c'}^{\text{op}}(\vec{k}, \omega') \rangle \\
= \sum_c \langle E_c^{\text{op}}(\vec{k}, \omega) \rangle \sum_c \langle E_{c'}^{\text{op}}(\vec{k}, \omega') \rangle + \sum_{\vec{k}} \langle E_c^{\text{op}}(\vec{k}, \omega) E_{c'}^{\text{op}}(-\vec{k}, \omega') \rangle
\]

mean field \quad \text{fluctuations}

(what you are interested in)

then use statistical independence + normalization to say

\[
\sum_c \langle \rangle \rightarrow \sum_s N_s \int f_s(\vec{\nu}) d\vec{\nu}
\]

\[
\sum A_i A_j = \sum_i A_i \sum_j A_j + \sum_i A_i^2
\]
Explain the meaning of the right hand side:

\[ \frac{\partial f_x}{\partial t} + \frac{\partial}{\partial x_\beta} (v_\beta f_x) + \frac{\partial}{\partial v_\beta} \left( \frac{f_{x\beta} f_x}{m_x} \right) = C_x [f_x] \]

How are the classical Langevin equations,

\[ \frac{d\vec{x}}{dt} = \vec{v} \]

\[ \frac{d\vec{v}}{dt} + \vec{v} = \delta a(t) \]

derived, and what physical effects do the terms represent?

In the Langevin Calculation, what is \( D_v \)?

How can this be related to the "typical step size" for \( \Delta v \) and \( \Delta t \)?
The right-hand side of the kinetic equation is the "collisional" term—it accounts for the rapidly fluctuating microfields & forces in the plasma, which arise when particles come close to each other.

\( f_e(r, \dot{r}, t) \) is a smoothed density averaged over a volume containing a large \# of particles. The force \( f_e \) is a smoothed macroscopic force and represents an average over time \& distance.

They came from Newton's laws, assuming acceleration due to:
1. Polarization drag: the \(-\dot{r} V^2\) term
2. Velocity space diffusion: random \( \dot{r} \) kicks from passing through Debye spheres. You can assume that \( \omega_p \ll \Delta t \ll \nu_e \) (coarse graining) and that Debye spheres are independent → then acceleration may be considered to be Gaussian white noise, \( \delta a(t) \) with \(<\delta a(t)\>) = 0 \<\delta a(t)\delta a(t')> = 2D_v \delta(t-t') \)

\[ D_v = \nu V^2 \text{ expect } D_v \sim \frac{\Delta V^2}{\Delta t} \text{ but be careful!} \]

\( V^2 \) and \( V_{th} \) are NOT \( \Delta V \) and \( \Delta t \)! Recall the process of scattering off of Debye spheres: \( \Delta t \sim \omega_p \)

- Setting \( \Delta V^2 \) gives \( \Delta V \sim V_{th} \)

→ Langevin calculation takes \( \epsilon \to 0 \) in just such a way that the velocity space diffusion coefficient remains finite while the elementary steps go to zero!
What is the Einstein relation, found from the Langevin calculation in a (B-field free) plasma?

Describe the Green's function technique for solving 1st order ODE's.

Describe the (unmagnetized) Langevin dynamics on short and long time scales.
Einstein relation is for $D_v$: find by using energy equipartition. Assume that at long times, a test particle has reached equilibrium with the background bath, such that

$$\frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} T$$

Langevin calc. gave $\langle \delta v^2 \rangle = D_v$ in long time limit.

Set equal to $\langle \delta v^2 \rangle = D_v = \frac{\nu_v^2}{\nu_{th}}$

Suppose you have a 1st order DE with a driving term

$$\hat{D} \Psi(t) = S(t).$$

You want to find a Green's function such that

$$\hat{D} \cdot G(t; \xi) = \delta(t - t')$$

Then the solution for $\Psi$ is given by:

$$\Psi(t) = \int_{-\infty}^{\infty} dt' G(t; t') S(t') \text{ initial condition}$$

$\text{Short times}$

$\nu t < 1$

$$\langle v \rangle = (1 - \nu^2) v_0 \text{ collisional slowing down}$$

$$\langle x \rangle = x_0 + v_0 t \text{ free streaming}$$

$$\langle \delta v^2 \rangle = 2 D v t \text{ velocity space diffusion}$$

$$\langle \delta x^2 \rangle = \frac{2}{3} D_0 v^2 t \text{ "orbit" diffusion}$$

$\text{Long times}$

$\nu t >> 1$

$$\langle v \rangle = 0 \text{ randomization of velocity}$$

$$\langle x \rangle = x_0 + \lambda_{\nu}$$

$$\langle \delta v^2 \rangle = \nu_v^2$$

$$\langle \delta x^2 \rangle = 2 D x t$$

$$D_v = \frac{\nu_v^2}{\nu_{th}}$$

$$D_x = \frac{\nu_v^2}{\nu_{th}}$$
what equation relates heat flux to a temperature gradient?
how do you estimate the coefficient?

what equation relates particle flux to a density gradient?
how do you estimate the coefficient?

explain the \( u \) dependent electron heat flux \( \tilde{q}_u \) in the unmagnetized, and magnetized cases.
\[ q = -k \frac{dT}{dx} \]

thermal conductivity

\[ k \sim n (\Delta x)^2 \frac{1}{\Delta t} \]

\[ \Gamma = -D \frac{dn}{dx} \]

diffusion coefficient:

\[ D \sim (\Delta x)^2 \frac{1}{2 (\Delta t)} \]

unmagnetized: faster electrons predominantly carry current (due to $\Delta n \Delta t$) → more fast electrons moving along $\vec{u}$, slower ones against. Though electron fluxes cancel, energy fluxes do not. Net energy flux in $\vec{u}$ direction (if $\vec{u}$ along $\vec{u}$)

Magnetized: electrons alternately accelerated/decelerated by friction force as they move along $\vec{u}$ against $\vec{B}$. Net effect: faster electrons moving in $\vec{q}$ → heat flux

\[ q \parallel \vec{u} \rightarrow \vec{B} \text{ and } \vec{u} \]
How does the $\bar{R}_{\Omega}$ force differ in the magnetized case from the unmagnetized case?

Derive a heuristic scaling for the temperature gradient dependent $\bar{R}_{\Omega}$ force. (unmagnetized case)

What is the physical mechanism for the $\bar{B}_T$ dependent $\bar{R}$ force? (unmagnetized case)
in the strongly magnetized case:

- the particles coming from regions of different temp (k) are moving in the y direction when the force imbalance occurs \Rightarrow force is now in z.
- typical distance of travel is now ρ, not λmp \Rightarrow magnitude of force: \delta R \sim \delta (m n V_i n V_e)
  \sim \frac{1}{\rho} \frac{\partial}{\partial x} (m n V_i n V_e)

previously, this was \sim \frac{1}{\lambda mp} magnitude differs by \frac{\rho}{\lambda mp}.

\bar{R} \sim m n V_i V_e \Rightarrow \delta \bar{R} \sim \delta (m n V_i n V_e)

\bar{V}_i \sim \frac{n e u}{T} \sqrt{\frac{T}{m}}

\frac{\partial}{\partial x} \left( \bar{V}_i \right) \sim \frac{1}{T^2} \frac{\partial T}{\partial x}

\frac{\partial}{\partial x} \left( m n V_i V_e \right) \sim \frac{\partial}{\partial x} \left( \frac{n e u}{T} \right)

\ldots \text{put all together:}

\delta \bar{R} \sim -n \frac{\partial T}{\partial x}

\bar{R} \rightarrow \tilde{R}

\tilde{R} \rightarrow \bar{R}

\tilde{R} \rightarrow x_{\tilde{R}} \rightarrow x

if particles arriving at x with same \bar{u}
(is no. \bar{R} is force) - consider those arriving from right + those from left.

\bullet \text{from right have higher energy, fewer collisions, smaller friction force.}

\Rightarrow \text{Net friction force opposite } \bar{R}.
How is the friction force related to $\Delta \tilde{u}_n$ and $\Delta \tilde{u}_+ \ $ in a strong magnetic field and why?

What equation relates momentum flux to a velocity gradient?

How do you estimate the coefficient?

Given the electric field of a single quasiparticle, $\tilde{E}_i^{\text{sp}}(k, \omega)$ in Fourier space, what is:

- the total electric field due to all quasiparticles?
- the two-time correlation function?
\[ \vec{R}_{\text{dif}} = -\frac{\partial E_{\text{ext}}}{\partial \vec{u}} \left( 0.51 \Delta u_m + \Delta u_4 \right) \]

friction force lower \( B \) field in \( z \) direction, due to \( B \) field prevents tail high energy electron tail (coeff \( \sim m \)) for shifted maxwellian

\[ \Pi_{xy} = -\eta \frac{\partial V_y}{\partial x} \]

viscosity, \( \eta \sim \frac{n m (\Delta x)^2}{\Delta t} \)

the total electric field due to all quasiparticles is:

\[ \vec{E}(k, \omega) = \sum_i \int \frac{d\omega}{2\pi} e^{-i\omega t} \vec{E}_{\text{ext}}(k, \omega) \]

and the two time correlation function is:

\[ \langle \vec{E}(k, \omega) \vec{E}^\ast(k', \omega') \rangle = \sum_i \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \langle \vec{E}_{\text{int}}(k, \omega) \vec{E}_{\text{int}}^\ast(k', \omega') \rangle \]
How do you calculate a dielectric function by inserting a test charge?

What is the answer for the Vlasov dielectric?

What is the Vlasov dielectric?

What does it reduce to in the high frequency limit?

How can the (scalar) conductivity be expressed in terms of the dielectric function?

Why shouldn't you evaluate or using the Vlasov dielectric for the hydrodynamic regime?
insert a test charge, \( \delta \rho^\text{test} \) related to a test potential by poisson:
\[
\nabla^2 \delta \varphi^\text{test} = -4\pi \delta \rho^\text{test}.
\]
Also know, \( \delta \varphi^\text{test} = \delta \varphi^\text{ind} + \delta \varphi^\text{test} \) (1)

use an eq. for the distribution function (like Vlasov or gyrokinetic eqn) \( \rightarrow E \) in it \( \rightarrow \) \( \mathbf{E}^\text{tot} = -\mathbf{k} \delta \varphi^\text{test} \)
linearize the eqn, solve for \( \mathbf{f}_i \) then get
\[
\delta \rho^\text{ind} = 2 \int f_i \, d\mathbf{v}
\]
use this and (1) to eliminate \( \delta \rho^\text{ind} \) to get relation between \( \delta \varphi^\text{test} \) and \( \delta \varphi^\text{test} \):
\[
\delta \rho^\text{test} = \frac{\delta \varphi^\text{test}}{\mathbf{D}(\mathbf{k}, \omega)} \quad \mathbf{D}(\mathbf{k}, \omega) = 1 + \sum_s \frac{4\pi^2 k_s^2}{k^2} \left( \int d\mathbf{v} \frac{\mathbf{k}_s \cdot \mathbf{v}}{\mathbf{w} - \mathbf{k} \cdot \mathbf{v}} \right)
\]

**Vlasov Dielectric:**
\[
\mathbf{D}(\mathbf{k}, \omega) = 1 + \sum_s \frac{\mathbf{w}_s^2}{k^2} \left[ \int d\mathbf{v} \frac{\mathbf{k}_s \cdot \mathbf{v}}{\mathbf{w} - \mathbf{k} \cdot \mathbf{v}} \right] = 1 + \sum_s \frac{\mathbf{w}_s^2}{k^2} \left[ \int d\mathbf{v} \frac{\mathbf{k}_s \cdot \mathbf{v}}{\mathbf{w} - \mathbf{k} \cdot \mathbf{v}} - \mathbf{w} \int d\mathbf{v} \delta(\mathbf{w} - \mathbf{k} \cdot \mathbf{v}) \frac{\mathbf{k}_s \cdot \mathbf{v}}{k^2} \right]
\]
for \( \omega \gg k v_{te} \)
\[
\approx 1 - \frac{\mathbf{w}_s^2}{\mathbf{w}^2} - \mathbf{w} \left( \sum_s \frac{\mathbf{w}_s^2}{k^2} F'(\mathbf{w}) \right)
\]

\[
\sigma(\mathbf{k}, \omega) = \frac{\mathbf{w}}{4\pi^2} \left[ \mathbf{D}(\mathbf{k}, \omega) - 1 \right]
\]

Don't use the Vlasov dielectric in the hydrodynamic (low frequency) regime because it was derived assuming high frequency regime - specifically, it's not ok to replace the resonant denominator \( (\omega - k \cdot \mathbf{V} + i\nu)^{-1} \) \( \rightarrow \) \( (\omega - k \cdot \mathbf{V} + i\nu)^{-1} \) for constant \( \nu \) because such a \( \nu \) does not respect the conservation properties of the true collision operation.
How do you use the Lorentz collision operator to find plasma conductivity?

Estimate the Dreicer limit for thermal runaway.

What is Boltzmann's H theorem?
write the kinetic equation:
\[
\frac{\partial f}{\partial t} + \nabla \cdot \mathbf{v} f + \mathbf{a} \cdot \nabla f = -\nabla [f]
\]
drop 1st 2 terms, taking "low $\lambda$, low $a$ limit".
Use Chapman-Enskog-like ordering, $f = f_0 + \epsilon f_1$ to get:
\[
\begin{align*}
\nabla [f_0] &= 0, \\
m \epsilon \frac{\partial f_0}{\partial t} &= -\nabla [f_1]
\end{align*}
\]
Use the Lorentz collision operator and solve for $f_1$.
Then use $\mathbf{u} = \int d\mathbf{v} \mathbf{v} f_1(\mathbf{v})$, and $\mathbf{j} = \mathbf{e} \times \mathbf{u}$.
get relationship between $f$ and $\mathbf{E}$.
Thus gives $\sigma$.

Deviator limit is field at which thermal particles may run away. Estimate by saying a particle will runaway if its thermal speed between collision times $\Rightarrow$
\[
V_{\text{te}} \frac{\mathbf{v}_i}{m} \leq \frac{3}{2} \epsilon \Rightarrow \epsilon < \frac{m}{c} V_{\text{te}} V_{\text{te}}
\]

Boltzmann's H theorem states that if a distribution function changes only by virtue of collisions, that no matter what the initial conditions, the distribution function must approach a Maxwellian in the course of time.

(approach of distribution function to Maxwellian by means of collisions is called 'relaxation'.)
What is the general form for the Fokker-Planck equation for the PDF of a variable $\beta$?

Make a table showing:
- velocity
- range of force
- duration of scattering event
- angular scatter in one event
- coupling parameter
- cross section for 90° scatter

for Boltzmann gas and the weakly coupled plasma.

How does the cross section for Rutherford scattering scale with $b_o$ and $\theta$? What happens at small angles?
\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial \beta} \left[ V(\beta, t) f(\beta, t) \right] - \frac{\partial^2}{\partial \beta^2} \left[ D(\beta, t) F(\beta, t) \right] = 0
\]

When \( V(\beta, t) = \frac{\Delta \beta}{\Delta t} \)

\[
D(\beta, t) = \lim_{\Delta t \to 0} \frac{\Delta \beta^2}{2 \Delta t}
\]

| \( V \) | Range \( V \) | Decay \( \gamma \) | Scattering event | Angle \( \theta \) per event | Coupling \( g^2 \) | \( t_{\text{int}} \) | \( t_{\text{int}} / \tau_0 \)
|-----------------|--------------|------------------|-------------------|---------------------------|-----------------|-------------------|
| Bethe-Mann        | \( V_{th} \) | \( b_0 \) | \( \frac{\Delta \theta}{\Delta t} \) | \( \sim 1 \) | \( \epsilon_n \) | \( n b_0 \) | \( \epsilon_n / n b_0 \)
| Dilute gas       | \( V_{th} \) | \( b_0 \) | \( \frac{\Delta \theta}{\Delta t} \) | \( \sim 1 \) | \( \epsilon_n \) | \( n b_0 \) | \( \epsilon_n / n b_0 \)

Weakly coupled plasma

\[
\sigma_R(\Theta) = \frac{b_0^2}{4 \sin^2 \left( \frac{\Theta}{2} \right)}
\]

At small angles \( \sigma_R \sim \frac{b_0^2}{\Theta^2} \rightarrow \) divergent for small angles.
What is the definition of a Markov process?

In the unmagnetized Langmuir calculations, name two ways in which you can "coarse grain" the time axis, and what you see in each.

What is the Klimontovich Eqn?

How does it differ from the Vlasov Eqn?
A sequence \( x_n \) of discrete random variables is Markov if the probability of observing \( x_n \) conditional on knowing the values of all the \( n-1 \) other variables depends in fact on just the value of \( x_{n-1} \).

\[ \Rightarrow \text{independent events} \Rightarrow \text{if present is known, future is independent of past.} \]

\[ \text{course-graining for diffusion in } \tau \ll \Delta t \ll \lambda \]

\[ \text{velocity space:} \]

\[ \text{course-graining for diffusion in } \lambda^{-1} \ll \Delta t \ll \text{traces} \]

\[ x\text{-space} \]

The Klimontovich eqn. looks just like the Vlasov eqn., but in \( N \) instead of \( f \):

\[ \tilde{N} (\mathbf{x}, t) = \frac{1}{N} \sum_{i=1}^{N} \delta (\mathbf{x} - \mathbf{z}(t)) \]

\[ \langle \tilde{N} \rangle = \langle \delta (\mathbf{z} - \mathbf{z}(t)) \rangle \text{ is one particle PDF} \]

\[ \nabla \cdot \tilde{N} = \text{Klimontovich includes all effects - collision, turbulence, etc., but is nonlinear in } \tilde{N} \]

because \( E \) depends on \( \tilde{N} \).

Vlasov eqn is from mean-field theory - contains no fluctuation effects (\( \epsilon \to 0 \)).
What is the Liouville eqn?
What is Liouville's Theorem?

What are the Bogoliubov time and spatial scales?
What happens on these scales?

What is this:
\[ \hat{C}_{0,1} \chi = -2\pi \left( \frac{n e^2}{\hbar m} \right) (n e^2) \ln \Lambda \frac{d}{d\nu} \right) + \mathbb{N} \mathbb{A} f_m \]
\[ \left\{ \left[ a(v)(\nabla \cdot \mathbf{v}) + b(v) \frac{d}{d\nu} \right] \left( \frac{1}{m} \frac{d\chi}{d\nu} \right) \right. \]
\[ - \left. \left( d^2 f_m(v) \mathbb{U}(\mathbf{v} \cdot \mathbf{v}) \left( \frac{1}{m} \frac{d\chi}{d\nu} \right) \right) \right\} \]

Where did it come from, and what physical effects correspond to terms 1, 4, and 6?
Liouville Eqn: \[ \frac{dP}{dt} + \nabla \times (\mathbf{V} \times P) = 0 \]

A continuity eqn:
System points never disappear.

Liouville's theorem states that phase space volume is conserved if \( \nabla \cdot \mathbf{V} = 0 \).
(incompressible flow)

\[ \tau_c \left( \frac{\omega_c}{\omega} \right) \ll \tau_c (\omega)^{-1} \ll \tau_v \]

\[ \lambda_D \ll \lambda_v \ll L \]

- Set up
- Debye
- Shelled

- Collision
- Time-velocity
- Space diffusion

- Hydrodynamic
- Time:
- Position
- Space
- Diffusion

It is the collision operator for like species collision, linearized around a Maxwellian distribution. \( f = f_0 + \chi \)

1. Which has \( \mathbf{v} \cdot \mathbf{V} \) describes pitch angle scattering
2. Which has \( \mathbf{v} \cdot \dot{\mathbf{V}} \) describes energy diffusion
3. Which has \( \int d\mathbf{v} \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{v}} \) ensures momentum conservation
What is the Multivariate Fokker-Planck equation?

What is the Chapman Kolmogorov Equation? What does it assume?

What is the Master Equation?
\[
\frac{df}{dt} + \frac{\partial}{\partial x} \left( \bar{v} f \right) - \frac{\partial^2}{\partial x^2} \left( D f \right) = 0
\]

where the drift and diffusion coefficients are:

\[
\bar{v} = \lim_{\Delta t \to 0} \frac{\langle \Delta x \rangle}{\Delta t} \quad D = \lim_{\Delta t \to 0} \frac{\langle \Delta x^2 \Delta \bar{v} \rangle}{2 \Delta t}
\]

\( \bar{x} \) is a vector of independent variables.

The Chapman-Kolmogorov Eqn gives transition probability for a Markov Sequence:

\[
f(n|s) = \int dr \ f(n|r) f(r|s)
\]

is integrate over all possible intermediate states. It is **NOT** exact, because it takes the probability of going from \( r \to n \) as being independent.

In general:

\[
f(n|s) = \int dr \ f(n|r,s) f(r|s)
\]

The Master Equation is the Chapman-Kolmogorov equation in the continuous time limit.
What is this:
\[ \frac{\partial f}{\partial t} + \nabla \cdot \mathbf{v} f = - C[f] \]

\[ C[f] = \frac{2}{3} \nabla \cdot \left[ \left( \frac{\mathbf{v} \cdot \hat{e}_p}{m} \right) f - \mathbf{D}(\mathbf{v}) \cdot \frac{\partial f}{\partial \mathbf{v}} \right] \]

and where did it come from?

What is the heat gained by electrons due to collisions with ions?

What is Hydrodynamics?
That is the general Fokker-Planck equation for an unmagnetized, weakly coupled plasma. It came from taking the jump moments for:

\[ \frac{d\mathbf{x}}{dt} = \mathbf{v} - \frac{d\mathbf{v}}{dt} = \delta \mathbf{x}(\mathbf{x}, t) + \left( \frac{q}{m} \right) \mathbf{E}\left(\mathbf{x}\right) \]

It is still general and \( E^{(n)} \) and \( D \) depend on fluctuation spectrum (which depends on \( f \)).

E field kick\(^\uparrow\) Polarization\(^\uparrow\)

in Debye sphere --- velocity spread\(^\dagger\)
diffusion.

\[ \mathbf{C} \mathbf{f} \mathbf{1} \] is the plasma Fokker-Planck collision operator minus sign

\[ \mathbf{Q} \mathbf{e} = - \mathbf{Q}_\mathbf{i} = \mathbf{R} \cdot \mathbf{U} \]

Energy exchange in Scattering (vanishes in Lorentz approximation) \( \rightarrow \) transfer of directed momentum into heat --- is finite in Lorentz approximation + contains ohmic heating.

Hydrodynamics is the study of the long wavelength, low frequency behavior of the plasma.
What are the linearized normal modes of the one component plasma?
What is the OCP?

What is the Plemelj Formula?

How do you show self-adjointness of a collision operator?
- Two shear modes
- One thermal diffusion mode
- Two plasma oscillations.

Note: the one component plasma is an electron fluid with a cold neutralizing ion background

\[
\frac{1}{\omega - k\nu \pm i\epsilon} = P\left( \frac{1}{\omega - k\nu} \right) + \delta(\omega - k\nu) \epsilon\nu
\]

- Want to work with the linearized operator \( \hat{C} \), where \( f = f_0(1 + \chi) \).

\( \left[ \hat{C} \right] = \hat{C} | \chi > \rightarrow \) find \( \chi \) \( \left[ \hat{C} \right] f = \left[ \hat{C} f \right] \)

To show self-adjointness, must show that

\[
\langle \psi | \hat{C} | \chi \rangle = \langle \chi | \hat{C} | \psi \rangle
\]

→ Use integration by parts
In conductivity, what is the Spitzer problem?

What are the solvability constraints in the Chapman-Enskog procedure?

What is the Chapman-Enskog ordering? When is it valid?
Spitzer problem is conductivity
in special case of zero frequency
and wave number.

- Write the first order eqn. in the form:
  \[ \langle \phi | \hat{C} | \psi \rangle = 0 \]
  then the
  solvability constraints amount to
  \[ \langle e_i | \psi \rangle = 0 \]
  where \( e_i \) are the null eigenfunctions of the
  adjoint operator \( \hat{A} \) for a self adjoint operator,
  the null eigenfunctions are all of the functions
  conserved by the collision operator. \( \Rightarrow \) there are as
  many solvability constraints as there are
  functions conserved by the collision operator

The Chapman-Enskog ordering is:

\[ \frac{1}{t} \frac{\partial}{\partial t} \sim O(\varepsilon) \ll 1 \quad \text{note: } \varepsilon \neq \varepsilon_0 \]

\[ \lambda_{mfp} \nabla \sim O(\varepsilon) \ll 1 \]

\[ \lambda_{mfp} \equiv \frac{V_{mfp}}{D} \]

It is valid in the Hydrodynamic regime.
How do you go about finding the Chapman-Enskog Equations?

Given a kinetic equation, how do you find the corresponding Langevin equations?

What is the expression for the friction force, $\mathbf{F}$?
- Use the Chapman-Enskog (hydrodynamic) ordering:
  \[ \frac{1}{2} \frac{\partial f}{\partial t} \sim O(\epsilon) \quad \frac{\partial}{\partial x} \frac{\partial f}{\partial x} \sim O(\epsilon) \]
  to put 1's in front of those terms in your eqn. Then write \( f = f_0 + \epsilon f_1 \Rightarrow \) solve the eqns. order by order.
  (likely to get \( C[f_0] = 0 \))

- Put the eqn. into Fokker-Planck form:
  \[ \frac{\partial}{\partial \mu} (V \nu) - \frac{\partial^2}{\partial \mu^2} (D \nu) = 0 \]
  Then general Langevin form is:
  \[ \frac{\partial \nu}{\partial t} + V + \delta a, \text{ where } \delta a \text{ satisfies:} \]
  \[ \langle \delta a(t) \delta a(t') \rangle = 2D \delta(t-t') \]

\[ \tilde{R} = \frac{\partial}{\partial \mu} (m \bar{v} e \bar{u}_e) \text{ coll} \]
\[ = - \int d \bar{v}_e (m \bar{n}) e \bar{v}_e C e i[f] \]