2006 Day 1 Question 5 (GPP)

a. $I = \frac{\partial}{\partial t}\left[\sum_j \int dv\, \frac{1}{2} m_j v^2 f_j + \frac{E^2+B^2}{8\pi}\right] + \nabla\cdot\left[\sum_j \int dv\, \frac{1}{2} m_j v^2 v f_j + \frac{c}{4\pi} E\times B\right]$

(1) (2) (3) (4)

$(2) = \partial_t\left(\frac{E^2+B^2}{8\pi}\right) = \frac{E\cdot\dot E + B\cdot \dot B}{4\pi} = \frac{c}{4\pi} E\cdot\nabla\times B - E\cdot J + B\cdot\dot B/4\pi$

$= \frac{c}{4\pi} B\cdot \cancel{\nabla\times E} - \frac{c}{4\pi}\nabla\cdot(E\times B) - E\cdot J + \cancel{B\cdot \dot B/4\pi}$

$(2)+(4) = -E\cdot J = -\sum q_j \int E\cdot v f_j\, dv$

$(1) = \sum_j \int dv\, \frac{1}{2} m_j v^2 \dot f_j = \sum_j \int dv\, \frac{1}{2} m_j v^2 \left(-\nabla\cdot(v f_j) - \nabla_v\cdot\left(\frac{F}{m} f_j\right)\right)$

$\left(\text{where } \frac{F}{m} = \frac{q_j}{m_j}\left(E + \frac{v\times B}{c}\right)\right)$

$(1)+(3) = -\sum_j \int dv\, \frac{1}{2} m_j v^2 \nabla_v\cdot(F/m\, f_j)$

$I = -\sum_j \left(\int dv\, \frac{1}{2} m_j v^2 \nabla_v\cdot\left(\frac{F}{m} f_j\right) + q_j v f_j\right)$

integrate by parts: $u = v^2$ $\quad dw = \nabla_v\cdot\left(\frac{F}{m} f_j\right)$

$du = 2v$ $\quad w = \frac{F}{m} f_j$

$I = -\sum_j q_j\left(\frac{1}{2} \frac{F}{m_j} f_j \Big|_{v=\pm\infty} - \int q_j v\cdot\left(E + \frac{v\times B}{c}\right) f_j + \int q_j v f_j \cdot E\, dv\right)$

since $v\cdot(v\times B) = 0$, the integrands cancel to 0.

Assuming $f_j \to 0$ as $v\to\infty$, the boundary term vanishes as well so this shows $I = 0$.

b. (1) kinetic energy density (3) kinetic energy flux

(2) electromagnetic energy density (4) EM flux (Poynting flux)

c. Integrate equation (6) over the chamber volume.

$\int d^3x\, \partial_t((1)+(2)) + \int d^3x\, \nabla\cdot((3)+(4)) = 0$

$\int d^3x\, \nabla\cdot(3) = \oint \sum_j \int d^3v\, \frac{1}{2} m_j v^2 (v\cdot da) \to 0$ since no plasma contacts the chamber wall

$\int d^3x\, \nabla\cdot(4) = \oint \frac{c}{4\pi}(E\times B)\cdot da = \oint \frac{c}{4\pi} B\cdot(\cancel{E\times da})^0$ conducting B.C.

This leaves only $\frac{d}{dt}\int d^3x\, (1)+(2) = 0$ as desired.