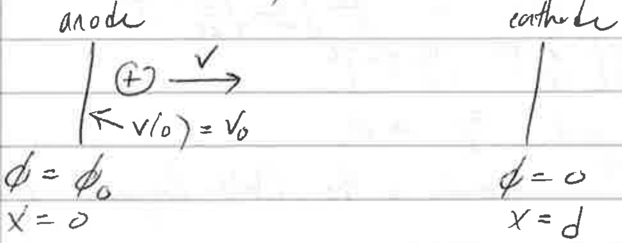
2008 Part 1, Question 1 (GPP1)

b.



$\nabla^2 \phi = -4\pi e n$ Poisson Equation

$\frac{1}{2} m v^2 + e\phi = E_0 = \frac{1}{2} m v_0^2 + e\phi_0$ energy conservation

$J_0 = e n v$

$$v = \sqrt{\frac{2e}{m}}\sqrt{\frac{E_0}{e} - \phi} = \sqrt{\frac{2e}{m}}\sqrt{\psi}$$

$$\phi'' = -\frac{4\pi J_0}{v} = -4\pi\sqrt{\frac{m}{2e}}\, J_0\, \psi^{-1/2}$$

$$\psi'' = 4\pi\sqrt{\frac{m}{2e}}\, J_0\, \psi^{-1/2} = a\psi^{-1/2}$$

$$\psi'\psi'' = a\psi'\psi^{-1/2}$$

$$\left(\tfrac{1}{2}\psi'^2\right)' = \left(2a\psi^{1/2}\right)'$$

$$\psi'^2 = 4a\psi^{1/2} + b$$

Note $\psi' = -\phi' \rightarrow \psi'^2 = \phi'^2$, so this is the form

$$\phi'^2 = \alpha(\phi_M - \phi)^{1/2} + \beta \quad \text{as desired}$$

where $\alpha = 4a = 8\pi\sqrt{\frac{m}{2e}}\, J_0$

$$\phi_M = \frac{E_0}{e} = \frac{m v_0^2}{2e} + \phi_0.$$

(?) c. $\beta = 0$ because $\beta < 0$ would prohibit the emission of ions with velocity v_0 (depending on magnitude of β they may still be emitted but with lesser initial velocity). $\beta > 0$ would draw more ions from the anode than are naturally emitted, leading to a piling up of charge until this E field was eliminated, reaching steady state and setting $\beta \rightarrow 0$

d. From above, $\psi'^2 = 4a\psi^{1/2}$

$$\int d\psi\, \psi^{-1/4} = \int\sqrt{4a}\, dx$$

$$\tfrac{4}{3}\psi^{3/4} = \sqrt{4a}\, x + c_1$$

$$\tfrac{4}{3}\psi^{3/4}(0) = \tfrac{4}{3}\left(\frac{m v_0^2}{2e}\right)^{3/4} \equiv \tfrac{4}{3}\left(\frac{K_0}{e}\right)^{3/4}$$

$$\psi^{3/4}(x) = \tfrac{1}{2}\cdot 3\sqrt{a}\, x + \left(\frac{K_0}{e}\right)^{3/4}$$

$$\psi^{3/4}(d) = \left(\frac{E_0}{e}\right)^{3/4} = \tfrac{3}{2}\sqrt{2\pi\sqrt{\frac{m}{2e}}\, J_0}\; d + \left(\frac{K_0}{e}\right)^{3/4}$$

$\rightarrow J_0 = \frac{1}{6\pi d}\sqrt{\frac{2e}{m}}\left[\left(\frac{E_0}{e}\right)^{3/4} - \left(\frac{K_0}{e}\right)^{3/4}\right]$.

~~In the $v_0 \rightarrow 0$ limit, $K_0 \rightarrow 0$ and $E_0 \rightarrow e\phi_0$, leaving~~

$$\cancel{\lim_{v_0 \rightarrow 0} J_0 = \frac{1}{6\pi d}\sqrt{\frac{2e}{m}}}$$

$\longrightarrow$

d. Previous page: $\left(\frac{E_0}{e}\right)^{3/4} - \left(\frac{K_0}{e}\right)^{3/4} = \frac{3d}{2}\sqrt{2\pi\sqrt{\frac{m}{2e}}\, J_0}$

$$J_0 = \frac{1}{2\pi}\sqrt{\frac{2e}{m}}\left(\frac{2}{3d}\right)^2\left[\left(\frac{E_0}{e}\right)^{3/4} - \left(\frac{K_0}{e}\right)^{3/4}\right]^2$$

in $V_0 \to 0$ limit, $K_0 \to 0$ and $\frac{E_0}{e} \to \phi_0$

$\lim_{V_0 \to 0} J_0 = (\quad)\, \phi_0^{3/2}$ which is the Child-Langmuir scaling.