2008 Part 1, Question 1 (GPP1)

a. \[ \text{anode} \quad \stackrel{+}{\rightarrow} \quad \text{cathode} \]
\[ \phi = \phi_0 \quad \phi = 0 \]
\[ x = 0 \quad x = d \]
\[ \nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \quad \text{Poisson Equation} \]
\[ \frac{1}{2} m v^2 + e \phi = E_0 = \frac{1}{2} m v_0^2 + e \phi_0 \quad \text{energy conservation} \]
\[ \frac{1}{\varepsilon_0} = \frac{e}{MV} \]
\[ v = \sqrt{\frac{2e}{m} \phi_0 / \varepsilon_0 - \phi} = \sqrt{\frac{2e}{m} J_0} \]
\[ \phi'' = -\frac{2\pi J_0}{\sqrt{2e}} - \frac{4\alpha}{\sqrt{2e}} \sum_n \sqrt{\frac{2e}{m} J_0 \tau_n^{-1/2}} \]
\[ n \tau_n \tau_n^{-1/2} = a \tau_n^{-1/2} \]
\[ n \tau_n = 4a\tau_n^{-1/2} + b \]

Note \( n \tau_n = -\phi' \rightarrow n \tau_n^2 = \phi' \]
so this is the form
\[ \phi' = x(2\phi - \phi_0) + \beta \quad \text{as desired} \]
where \( x = 4a = \frac{8\pi}{\sqrt{2e}} \sum_n \frac{J_0}{\tau_n} \]
\[ \phi_0 = \frac{\phi_0}{e} = \frac{mv_0^2}{2e} + \phi_0. \]

b. \( B = 0 \) because \( B \neq 0 \) would prohibit the emission of ions with velocity \( v_0 \). Depending on magnitude of \( B \), they may still be emitted but with lesser initial velocity. \( B > 0 \) would draw more ions from the anode than are naturally emitted, leading to a higher up to 0 charge until this E-field was eliminated, reaching steady state and setting \( B \to 0 \).

d. From above,
\[ n \tau_n^{-1/2} = \sqrt{4a} \quad \tau_n \to \frac{1}{6} \frac{2e}{m} \left( \frac{\phi_0}{e} \right)^{3/4} \]
\[ \frac{1}{3} \tau_n^2 = \sqrt{4a} x + \frac{C}{3} \]
\[ \frac{3}{2} \tau_n^2 (0) = \frac{3}{2} \left( \frac{mv_0^2}{2e} \right)^{3/4} \equiv \frac{3}{2} \left( \frac{K_0}{e} \right)^{3/4} \]
\[ \tau_n^2 (x) = \frac{3}{2} \sqrt{4a} x + \left( \frac{K_0}{e} \right)^{3/4} \]
\[ \tau_n^2 (0) = \left( \frac{E_0}{e} \right)^{3/4} = \frac{3}{2} \sqrt{\frac{m}{2e}} \tau_n \text{d} + \left( \frac{K_0}{e} \right)^{3/4} \]
d. Previous page: \[ \left( \frac{E_0}{e} \right)^{3/4} - \left( \frac{K_0}{e} \right)^{3/4} = \frac{3d}{2} \sqrt{2\pi \sqrt{\frac{m}{2e}}} J_0 \]

\[ J_0 = \frac{1}{2\pi} \sqrt{\frac{2e}{m}} \left[ \left( \frac{E_0}{e} \right)^{3/4} - \left( \frac{K_0}{e} \right)^{3/4} \right]^2 \]

\[ \text{in } V_0 \to 0 \text{ limit, } K_0 \to 0 \text{ and } \frac{E_0}{e} \to \phi_0 \]

\[ \lim_{V_0 \to 0} \frac{J_0}{e^{3/4}} = \left( \frac{3d}{2} \right) \phi_0^{3/2} \text{ which is the Child-Langmuir scaling.} \]