2009 Day 2 3A (math)

$y'' - xy' - 3y = 0$

a. $x \to 0$ ordinary solution $\to y = \sum a_n x^n$

$\sum a_n n(n-1) x^{n-2} - \sum (3a_n + n) a_n x^n$

$n - 2 = j \Leftrightarrow j + 2 = n$

$\sum_{j=n-2} a_{j+2} (j+2)(j+1) x^j$

$\sum_{n \geq 0} a_{n+2}(n+2)(n+1) - (n+3) a_n \; x^n = 0$

$a_{n+2} = \dfrac{n+3}{(n+2)(n+1)} a_n \longrightarrow$ even + odd soln.

b. $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$, $y'' = \dot{y}\frac{d^2t}{dx^2} + \ddot{y}\left(\frac{dt}{dx}\right)^2$

$x = \frac{1}{t}$, $dx = \frac{-1}{t^2}dt$, $\frac{dt}{dx} = -\frac{1}{x^2}$, $\frac{d^2t}{dx^2} = \frac{2}{x^3} = 2t^3$

$t^4 \ddot{y} + 2t^3 \dot{y} - \frac{1}{t}(-t^2)\dot{y} - 3y = 0 \longrightarrow t \to 0$ irregular.

$S'' + S'^2 - xS' - 3 \sim 0$

$S'^2 - xS' - 3 \sim 0$ $\quad \frac{S''}{S'^2} \ll 1$

1. $S'(S' - x) \sim 0 \to S' \sim x$ works
2. $S'^2 \sim 3$ weak
3. $S' \sim \frac{-3}{x}$ ✓

balanced

$f'' + g'' + \cancel{f'^2} + g'^2 + 2f'g' - \cancel{xf'} - xg' - 3 \sim 0$

$1 + g'' + g'^2 + 2xg' - xg' - 3 \sim 0$ $\quad g'' \ll g'^2,\ g' \ll f'$

$g'^2 + xg' - 2 \sim 0$

1. $g'(g' + x) \sim 0$ fails $g' \ll f'$
2. $g' \sim \pm\sqrt{2}$ weak
3. $g' \sim \frac{2}{x}$ ✓ $\to y \sim e^{\frac{1}{2}x^2 + \log x^2} = x^2 e^{\frac{1}{2}x^2}$

$\frac{3}{x^2} + g'' + \frac{9}{x^2} + g'^2 - \frac{6}{x}g' - xg' \sim 0$

$g'^2 - xg' + \frac{12}{x^2} \sim 0$

1. $g'(g' - x) \sim 0$ not $g' \ll f'$
2. $g' \sim \pm i\frac{\sqrt{12}}{x}$ weak
3. $g' \sim \frac{12}{x^3}$ ✓ $\to y \sim e^{-3\log x - \frac{6}{x^2}} = \frac{1}{x^3} e^{-6/x^2}$

e. $y(x) = \int_C e^{xt} f(t)\,dt$

$\int_C (t^2 - xt - 3)\, e^{xt} f(t)\,dt = 0$

$u = t f(t) \qquad dv = x e^{xt}$

$du = f + t f' \qquad v = e^{xt}$

$t f(t) e^{xt}\Big|_C + \int_C \left(t^2 f(t) + t f'(t) - 2 f(t) \right) e^{xt}\,dt = 0$

$(t^2 - 2) f = -t f' \to \int \frac{df}{f} = -\int \frac{t^2-2}{t}\,dt$

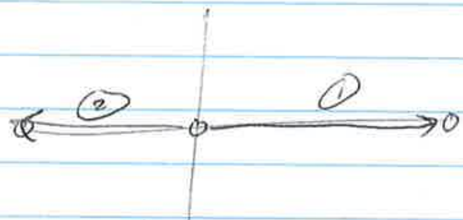
$\log f = C \; ~~\ln~~ - \frac{1}{2} t^2 + 2 \log t$

$f = A t^2 e^{-\frac{1}{2} t^2}$ verify $f' = 2t e^{-\frac{1}{2}t^2} - t^3 e^{-\frac{1}{2}t^2}$

$= \frac{2f}{t} - tf$ ✓.

$t^3 e^{-\frac{1}{2}t^2} e^{xt}\Big|_C \longrightarrow 0$ at $t = 0$, $t \to \pm\infty$

$y(0) = \int_C A t^2 e^{-\frac{1}{2}t^2}\,dt = 1$



$y(x \to \infty) \int_C A t^2 e^{-\frac{1}{2}t^2} e^{xt} \longrightarrow 0$

must have $C: 0 \to \infty$.

$\int_0^\infty t^2 e^{-\frac{1}{2}t^2}\,dt = \frac{1}{A}$

$v = \sqrt{\frac{1}{2}}\, t$

$= \int_0^\infty 2 v^2 e^{-v^2}\, d(\sqrt{2} v) = 2\sqrt{2} \left(-\frac{\partial}{\partial a}\right) \int_0^\infty e^{-a v^2} \Big|_{a=1}$

$= 2\sqrt{2} \left(-\frac{\partial}{\partial a}\right) \frac{1}{2}\sqrt{\frac{\pi}{a}} = \sqrt{2}\,\frac{1}{2}\sqrt{\pi} \to A = \frac{2}{\sqrt{2\pi}}$

$y(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty t^2 e^{-\frac{1}{2}t^2} e^{xt}\,dt$