2009, Day 1, Q 1 (Irreversibles)

a. Taylor Formula $D = \lim_{t\to\infty}\int_0^t C_{vv}(\tau)\,d\tau$

Derivation: $D \equiv \lim_{t\to\infty} \frac{\langle \delta x^2(t)\rangle}{2t}$

$$\langle x(t)\rangle = x_0 + \underbrace{\int_0^t dt' \langle v(t')\rangle}_{\delta x(t)}$$

$$\langle \delta x(t')\delta x(t'')\rangle = \int_0^t\int_0^t dt'\,dt'' \langle \delta v(t')\delta v(t'')\rangle = \int_0^t\int_0^t dt'\,dt''\, C_{vv}(t',t'')$$

If stationary, $C_{vv}(t',t'') = C_{vv}(t'-t'') = C_{vv}(\tau)$

Switch order of integration with sum and difference variables:

$$\langle \delta x(\tau)\delta x(0)\rangle = 2t\int_0^t d\tau\, C_{vv}(\tau)\left(1-\frac{\tau}{t}\right) \approx 2t\int_0^t d\tau\, C_{vv}(\tau)$$

b. $$D = v_0^2\int_0^\infty \cos(\omega_0\tau)\, e^{-\nu|\tau|}\, d\tau = \mathrm{Re}\, v_0^2 \int_0^\infty e^{(i\omega_0-\nu)\tau}\, d\tau$$

$$= v_0^2\, \mathrm{Re}\, \frac{1}{i\omega_0-\nu}\left(e^{(i\omega_0-\nu)\infty} - 1\right) = v_0^2\, \mathrm{Re}\, \frac{\nu+i\omega_0}{\nu^2+\omega_0^2} = \frac{v_0^2\,\nu}{\nu^2+\omega_0^2}$$

(assuming $\nu > 0$).

2009 Day 1 Q1 (Irreversibles)

a. Taylor Formula $D_x = \int_0^{\infty} C_{vv}(\tau)\, d\tau$

b. $D = v_0^2 \int_0^{\infty} \cos(\omega t)\, e^{-\nu t} = v_0^2 \,\mathrm{Re} \int_0^{\infty} e^{(i\omega - \nu)t}\, dt$, $\nu > 0$

$= \mathrm{Re}\, v_0^2 \frac{1}{i\omega - \nu} e^{(i\omega-\nu)t} \Big|_{t=0}^{\infty} = \mathrm{Re}\, v_0^2 \frac{1}{\nu - i\omega} = v_0^2 \frac{\nu}{\nu^2 + \omega^2}$