2010 Day 2 Question 2 (EPP)

a. $n = \int_{-\infty}^{\infty} dv\, f = A(V_+ - V_-)$

$nV = \int v f = \frac{1}{2} A (V_+^2 - V_-^2)$

$P = m\int (v-V)^2 f = m\int v^2 f - 2mV\int v f + mV^2 \int f$

note $V = \dfrac{\frac{1}{2} A (V_+^2 - V_-^2)}{A(V_+ - V_-)} = \frac{1}{2}(V_+ + V_-)$.

$P = Am\left[\frac{1}{3}\cancel{A}(V_+^3 - V_-^3) - \frac{1}{2}(V_+^2 - V_-^2)(V_+ + V_-) + \frac{1}{4}(V_+ + V_-)^2 (V_+ - V_-)\right]$

$= \frac{mA}{12}\left[4(V_+^3 - V_-^3) - 6(V_+ + V_-)^2(V_+ - V_-) + 3(V_+ + V_-)^2(V_+ - V_-)\right]$

$\left[4(V_+^3 - V_-^3) - 3(V_+ + V_-)^2(V_+ - V_-)\right]$

$\left[V_+^3 - V_-^3 - 3V_+^2 V_- + 3V_+ V_-^2\right]$

$= \frac{mA}{12}\left[V_+ - V_-\right]^3$

b. $(\partial_t + v\partial_x - \epsilon\partial_v) f = 0 \qquad \epsilon = \frac{e}{m} E$

0-moment: $\partial_t \int f + \partial_x \int v f - \epsilon \int \partial_v f = 0$

$A\partial_t (V_+ - V_-) + A\frac{1}{2}(V_+^2 - V_-^2) = 0$

1-moment: $\partial_t \int v f + \partial_x \int v^2 f - \epsilon \int v\, \partial_v f = 0$

Note $f = A\,\Theta(v - V_-) - A\,\Theta(V_+ - V_m)$

$\partial_v f = A\,\delta(v - V_-) - A\,\delta(V_+ - v)$

$A\frac{1}{2}(V_+^2 - V_-^2) + \frac{1}{3}A(V_+^3 - V_-^3) - \epsilon A (V_+ - V_-) = 0$

c. $V_+ \dot{V}_+ + V_+^2 V_+' + \epsilon V_+ - (V_- \dot{V}_- + V_-^2 V_-' + \epsilon V_-) = 0$

$V_+ (\dot{V}_+ + V_+ V_+' + \epsilon) - V_-(\dot{V}_- + V_- V_-' + \epsilon) = 0$

By (7), $\dot{V}_+ + V_+ V_+' + \epsilon = \dot{V}_- + V_- V_-' + \epsilon$, so

$(V_+ - V_-)\begin{Bmatrix} \dot{V}_+ + V_+ V_+' + \epsilon \\ \dot{V}_- + V_- V_-' + \epsilon \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

This gives the desired equations since in general $V_+ \neq V_-$:

$\dot{V}_\pm + V_\pm V_\pm' = -\epsilon$

$\longrightarrow$

d. $\int (v-V)^3 f = \int v^3 - 3v^2V + 3vV^2 - V^3 f \, dv$

$\int v^3 f = \frac{1}{4} A (V_+^4 - V_-^4)$

$-3V\int v^2 f = -\frac{1}{2} A (V_+ + V_-)(V_+^3 - V_-^3)$

$3V^2 \int v f = \frac{3}{8} A (V_+ + V_-)^2 (V_+^2 - V_-^2)$

$-V^3 \int f = -\frac{1}{8} A (V_+ + V_-)^3 (V_+ - V_-)$

Verify these add to zero. Let $x = V_+$, $y = V_-$:

$2(x^4 - y^4) - 4(x^3-y^3)(x+y) + 3(x^2-y^2)(x+y)^2 - (x-y)(x+y)^3$

$2(x^2+y^2)(x^2-y^2)$

$2(x^2+y^2)(x+y)(x-y)$

→ $2(x^2+y^2)(x+y) - 4(x^3-y^3) + 3(x+y)^2(x-y) - (x+y)^2(x-y)$

$(x^2+y^2)(x+y) - 2(x^3-y^3) + (x+y)^2(x-y)$

$\cancel{(x^2+y^2) - 2(x^2-xy+y^2) + (x^2-y^2)}$

expand $(x^3+y^3+xy^2+x^2y) + (-2x^3+2y^3) + (x^2+2xy+y^2)(x-y)$

$\cancel{(x^3-y^3+x^2y-xy^2)}$

$2(x^2-y^2)(x^2$

$2(x^4-y^4) - 4(x^3-y^3)(x+y) + 3(x^2-y^2)(x+y)^2 - (x-y)(x+y)^3 \overset{?}{=} 0$

Factor x^2-y^2 from each term

$x^2+y^2 - 2(x^2+xy+y^2) + (x+y)^2 \overset{?}{=} 0$

Expanding shows this to be true.

Or via clever integration instead of clever factoring:

$I = \int (v-V)^3 f \, dv$, integrate by parts

$= (v-V)^4 f \Big|_{-\infty}^{\infty} - \int 4(v-V)^3 f + (v-V)^4 f'$

Surface term → 0 (otherwise big distribution not normalizable)

$5I = -\int (v-V)^4 f' \, dv$ where $f' = A(\delta(v-V_-) - \delta(V_+-V_-))$

$= -A[(V_- - V)^4 - (V_+ - V)^4]$ where $V = \frac{1}{2}(V_+ + V_-)$

$= -A([\frac{1}{2}(V_- - V_+)]^4 - [\frac{1}{2}(V_+ - V_-)]^4) = 0.$