2010 Day 1 Question 5B (Asymptotics)

1. $y'' + xy' + 2y = 0 \qquad x \to 0$ ordinary point.

$y = \sum a_n x^n$

$\sum n(n-1)a_n x^{n-2} + \sum (n a_n + 2a_n) x^n = 0$

$j = n-2 \iff n = j+2$

$\sum_0 \left[(n+2)(n+1)a_{n+2} + (n+2)a_n\right] x^n = 0$

$a_{n+2} = -\dfrac{a_n}{n+1}$ use even, odd powers.

2. $x = \frac{1}{t}, \quad \frac{dx}{dt} = \frac{-1}{t^2}, \quad \frac{dt}{dx} = -\frac{1}{x^2}, \quad \frac{d^2t}{dx^2} = 2t^3$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}, \quad \frac{d^2y}{dx^2} = \ddot{y}\left(\frac{dt}{dx}\right)^2 + \dot{y}\frac{d^2t}{dx^2}$

$t^4\ddot{y} + 2t^3\dot{y} - \frac{1}{t}(t^2)\dot{y} + 2y = 0 \to$ irregular. $\quad y = e^{S(x)}$

$S'' + S'^2 + xS' + 2 \sim 0$

$S'^2 + xS' + 2 \sim 0$

1. $S'(S' + x) \sim 0 \to S' \sim -x$ ✓
2. $S'^2 + 2 \sim 0 \to S' \sim \pm i\sqrt{2}$ weak
3. $xS' + 2 \sim 0 \to S' \sim \frac{-2}{x}$ ✓

$f'' + g'' + \cancel{f'^2} + g'^2 + 2f'g' + \cancel{xf'} + xg' + 2 \sim 0$

$-1 + g'^2 - 2xg' + xg' + 2 \sim 0$

$g'^2 - xg' + 1 \sim 0$

1. $g'(g' - x) \sim 0$ not $\ll f'$
2. $g'^2 \sim \pm i$ weak
3. $g' \sim \frac{1}{x}$ ✓ $\longrightarrow y_1 \sim x e^{-\frac{1}{2}x^2}$

$f'' + g'' + f'^2 + g'^2 + 2f'g' + \cancel{xf' + xg' + 2} \sim 0$

$\frac{2}{x^2} + \frac{4}{x^2} + g'' + g'^2 - \frac{4}{x}g' + xg' \sim 0$

$g'^2 + xg' + \frac{6}{x^2} \sim 0$

1. $g'(g' + x) \sim 0$ not $\ll f'$
2. $g' \sim \pm i\sqrt{6}/x$ weak
3. $g' \sim -\frac{6}{x^3}$ ✓ $\to y_2 \sim \frac{1}{x^2} e^{3/x^2}$

3. $y'' + xy' + 2y = 0$, $\quad y(x) = \int_C e^{xt} f(t)\,dt$

$\int_C (t^2 + xt + 2) f(t)\, e^{xt}\,dt = 0$

$u = tf \qquad dv = x e^{xt}$

$du = tf' + f \qquad v = e^{xt}$

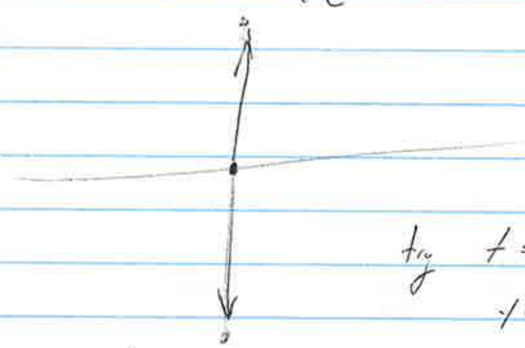
$\int_C \left[(t^2+1) f(t) - t f'(t) \right] e^{xt}\,dt + t f(t) e^{xt} \Big|_C = 0$

$\int \frac{df}{f} = \int t + \frac{1}{t}\,dt$

$\log f = \frac{1}{2}t^2 + \log t + c$

$f = A t e^{\frac{1}{2}t^2} \qquad \left(f' = e^{\frac{1}{2}t^2} + t^2 e^{\frac{1}{2}t^2} = (t^2+1)\frac{f}{t} \right)$

$\rightarrow \quad A t^2 e^{\frac{1}{2}t^2 + xt} \Big|_C = 0 \quad$ as $t \to 0$, $t \to \pm i\infty$



try $t = iy$, $dt = i\,dy$

$y(x) = \int_{-\infty}^{\infty} (iy)\, e^{-\frac{1}{2}y^2} (i\,dy) = -\int_{-\infty}^{\infty} y e^{-\frac{1}{2}y^2}\,dy = 0.$

4. $y(0) = A \int_C t e^{\frac{1}{2}t^2}\,dt$

$y'(0) = A \int_C t^2 e^{\frac{1}{2}t^2}\,dt \qquad$ works too for normalizing choice of A.

$x \to \infty$, $f(x) = -A \int_{-\infty}^{\infty} t e^{xt + \frac{1}{2}t^2}\,dt$

$\phi(x,t) = xt + \frac{1}{2}t^2$